Methods of constructing quantum principal bundles.

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Methods of Constructing Quantum Principal Bundles

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Preface

The history of Hopf-Galois extensions stretches back to the works of Chase and Sweedler ([18]) and Kreimer and Takeuchi ([29]). The structure of Hopf-Galois extensions including their appearance in context of quantum homogeneous spaces was extensively discussed in the works by Schneider ([38] and [39]). The geometric significance of Hopf-Galois extensions was fully realised in [9], where they were identified as the dualisation of the classical notion of principal bundles. A quantum gauge theory, i.e, quantum gauge transformations, connections, curvature, etc., was proposed in [9], and later developed in [23]. In particular, in [23] the notion of a strong connection was introduced.

The development of theory of coalgebra Galois extensions, which generalise the theory of Hopf-Galois extensions, was motivated by the emergence of some natural examples of quantum homogeneous spaces, which were not Hopf-Galois extensions. General coalgebra-Galois extensions were introduced in [6], following the theory of principal coalgebra bundles introduced and developed in [10], in the framework of entwining structures. The theory of strong connections, including a non-trivial example of non-commutative Hopf fibration, was extended to coalgebra bundles in [11].

Coalgebra-Galois extensions, interpreted as quantum principal bundles, are a very important type of quantum spaces. Construction of new examples is therefore crucial for further development of noncommutative geometry.

In various areas of mathematics, an important part of any theory is constituted by methods of constructing new structures by combining the existing ones, as well as by theorems which state how the properties of the combined structure derive from the properties of its components.

In our dissertation we discuss two methods of constructing new quantum principal bundles by combining the existing ones: cotensor product and gluing. In addition, these methods, especially the latter, provide insight into the structure of many of the existing coalgebra-Galois extensions.

Our dissertation is organised as follows:

Chapter 1 contains a very short introduction to the concepts of noncommutative geometry and its basic mathematical methods, which will be used later. Topics covered include the Gelfand-Naimark Theorem, entwining structures, projective modules and differential calculuses.

Chapter 2 introduces coalgebra-Galois extensions (quantum principal bundles), strong connections and associated bundles. Section 2.4, based on the joint paper [5] with T. Brzeziński and L. Dąbrowski, gives an example of a quantum Hopf fibration. A strong connection form and projectors for associated line bundles of any charge are explicitly constructed, ilustrating methods covered by the earlier part of the chapter.

Chapter 3 is devoted to cotensor products of coalgebra and Hopf-Galois extensions. The main results are Theorem 3.2.5, which states the conditions implying that the cotensor product of two quantum principal bundles is again a coalgebra-Galois extension, and Theorem 3.4.8 and Theorem 3.4.9, which consider existence of a strong connection on the cotensor product. These results are illustrated by examples in Section 3.3 and Section 3.5, where we consider two different ways of cotensoring together two copies of Matsumo sphere. The chapter is closed with remarks on cotensor products of cleft extensions in Section 3.6.

Chapter 4 contains the results of our research on locally coalgebra-Galois extensions. Sections 4.2 and 4.3 recall the relevant results from [15] on covering and gluing of modules. In Section 4.4 the concept of a locally coalgebra-Galois extension is introduced, and the conditions which imply that a locally coalgebra-Galois extension is a (global) coalgebra-Galois extension are considered. The main results of this section are Theorem 4.4.8 and Corollary 4.4.10. Section 4.5 introduces locally cleft extensions, which can be considered as the direct generalisation of locally trivial principal bundles defined in [15]. Many results of this section seem to be reformulations, in terms of cleft extensions, of the results from [15], which, nevertheless, involve considering nontrivial conditions, particular to cleft, nontrivial extensions. The main result of this section, is Proposition 4.5.9. Section 4.6 describes gluing of cleft extensions. The methods described in this chapter are illustrated, in Section 4.7, by explicit construction of quantum lens spaces by gluing of two quantum solid tori.

Contents

Acknowledgements iii

Preface v

1 Introduction 1

1.1 C^*-algebras and geometry . . . 1

1.2 Examples . . . 2

1.2.1 The quantum unit disc . . . 2

1.2.2 The quantum torus . . . 5

1.3 Modules and comodules . . . 6

1.3.1 Basic notation and terminology . . . 6

1.3.2 Noncommutative geometry and basic algebraic structures . . . 7

1.4 Projective modules . . . 9

1.5 Entwining structures . . . 10

1.6 Differential calculus . . . 13

2 Quantum principal bundles 15

2.1 Introduction . . . 15

2.2 Strong connections . . . 18

2.3 Quantum vector bundles and associated bundles . . . 21

2.3.1 Vector bundles . . . 21

2.3.2 Associated vector bundles . . . 22

2.4 Example: A noncommutative Hopf fibration . . . 24

2.4.1 Introduction . . . 24

2.4.2 A quantum Hopf fibration . . . 25

2.4.3 Monopole connection and projectors of charge n . . . 28

2.4.4 Remarks about adjoining elements and representations . . . 31

2.5 Cleft extensions . . . 32

2.5.1 The quantum solid torus . . . 34

3 Products of quantum bundles 37

3.1 Introduction . . . 37

3.2 The inverse of the canonical map . . . 38

3.3 Example: The cotensor product of Matsumoto spheres I . . . 47

3.4 Strong connections and cotensor products . . . 49

3.5 Example: The cotensor product of Matsumoto spheres II 58
3.6 Cotensor products and cleft extensions 64

4 Locally C-coalgebra Galois extensions **69**
4.1 Introduction 69
4.2 Covering of modules and algebras 70
4.3 Gluing of modules and algebras 72
4.4 Locally C-coalgebra Galois extensions 73
4.5 Locally cleft extensions 84
4.6 Gluing cleft extensions 92
4.7 Example: The quantum lens spaces 95
4.7.1 Gluing of two quantum solid tori 96
4.7.2 Lens spaces of positive charge 100
4.7.3 The inverse of the canonical map on $\vartheta(L_\beta^{p,q,\theta})$ 103
4.7.4 Representations of $\vartheta(L_\beta^{p,q,\theta})$ 104
4.7.5 Final remarks 106

Chapter 1

Introduction

Noncommutative geometry, as envisaged by Connes ([19]), considers functions on spaces with values in $\mathbb{C}$ and having the structure of a (not necessarily commutative) C^*-algebra, as the primitive building blocks of geometry, instead of points. Due to the Gelfand-Naimark Theorem, this contains as a special case the geometry of the locally compact Hausdorff topological spaces.

Due to the extensive developments in the research of the purely algebraic, geometry inspired, structures the term 'noncommutative geometry' as used in this work refers to the algebraic aspects of Connes' noncommutative geometry, where we weaken assumption about the functions on the noncommutative spaces, assuming only that they form a $\mathbb{K}$-algebra, where $\mathbb{K}$ is a commutative unital ring.

The aim of this chapter is to recall the basic algebraic and analytical notions used in the subsequent part of this work, and as to give their interpretation in terms of noncommutative geometry.

1.1 C^*-algebras and geometry

Let V be a locally convex topological vector space (real or complex) and let V^* be the set of continuous linear functionals on V. The family of seminorms

$$\{v \mapsto |f(v)| : V \to \mathbb{R} \mid f \in V^*\}$$

defines a *weak topology* on V. Similarly, the family of seminorms

$$\{f \mapsto |f(v)| : V^* \to \mathbb{R} \mid v \in V\}$$

defines a *weak* topology* on V^*.

A $\mathbb{C}$-algebra endowed with an involutive, antilinear and antialgebra map $* : A \to A$ is called a *$*$-algebra*.

A $\mathbb{C}$-algebra A, such that A is also a Banach space with norm $\|\cdot\|$ which is submultiplicative, i.e., $\|aa'\| \leq \|a\|\|a'\|$, for all $a, a' \in A$, is called a *Banach algebra*. In addition, if A is unital, then A is called a *unital Banach algebra* if $\|1_A\| = 1$. For all $a \in A$, the set

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid a - \lambda \text{ is not invertible } \} \tag{1.1}$$

is called the *spectrum* of a.

A $*$-algebra A which is also a Banach algebra, such that $\|a^*a\| = \|a\|^2$ is called a *C^*-algebra.*

Let X be a locally compact Hausdorff topological space. We shall denote by $C_0(X)$ the set of continuous functions $f : X \to \mathbb{C}$ vanishing at infinity, i.e., such that, for all $\epsilon > 0$, the set $\{x \in X | |f(x)| > \epsilon\}$ is compact. $C_0(X)$ is naturally a C^*-algebra with the pointwise multiplication and the pointwise complex conjugation as $*$ operation, and with a supremum norm, i.e.,

$$\|f\|_\infty = \sup_{x \in X} |f(x)|,$$

for all $f \in C_0(X)$.

A *character* on any $\mathbb{C}$-algebra A is a nonzero algebra homomorphism $\tau : A \to \mathbb{C}$. We denote $\Omega(A)$ the set of characters on A. Suppose A is an abelian Banach algebra. Then characters are bounded linear operators (Theorem 1.3.4 [32]). Therefore one can endow the space $\Omega(A)$ with a relative weak* topology. It turns out (Theorem 1.3.5 [32]) that $\Omega(A)$ is a locally compact Hausdorff space. If A is unital, then $\Omega(A)$ is compact.

For all $a \in A$, the function

$$\hat{a} : \Omega(A) \to \mathbb{C}, \quad \tau \mapsto \tau(a), \tag{1.2}$$

called a *Gelfand transform of a,* is continuous and vanishes at infinity, i.e. $\hat{a} \in C_0(\Omega(A))$. The map

$$a \mapsto \hat{a} : A \to C_0(\Omega(A)) \tag{1.3}$$

is a norm decreasing homomorphism (Theorem 1.3.6. [32]), and furthermore, if A is an abelian C^*-algebra, it is an isometric $*$-isomorphism (Theorem 2.1.10 /Gelfand/ [32]).

On the other hand, it is obvious that, for all $x \in X$, the map

$$\tilde{x} : C_0(X) \to \mathbb{C}, \ \ f \mapsto f(x)$$

is a character on $C_0(X)$. In fact it can be proven (Theorem 2.1.15 [32]) that, if X is compact, then all the characters on $C_0(X)$ are of this form. Moreover, if $C_0(X)$ is isomorphic to $C_0(Y)$ as a C^* algebra, then X and Y are homeomorphic (for any locally compact topological Hausdorff spaces X and Y). In this way, we have established a bijective correspondence between isomorphism classes of commutative C^* algebras and homeomorphism classes of locally compact Hausdorff topological spaces.

1.2 Examples

1.2.1 The quantum unit disc

A two-parameter family of quantum unit discs was defined in [28]. Here we consider the one parameter subfamily studied in [27]. We start with a coordinate $*$-algebra $\vartheta(D_p)$ generated by x and the relation

$$x^*x - pxx^* = 1 - p, \ \ 0 < p < 1. \tag{1.4}$$

The spectrum of xx^* (in any C^*-algebra completion) is

$$\sigma(xx^*) = \{1 - p^n \mid n = 0, 1, 2, \ldots\} \cup \{1\}, \tag{1.5}$$

i.e. $xx^* \leq 1$, where $\leq$ is understood as an order relation between positive operators. This justifies the name 'unit disc'. Furthermore, this relation means also that $\|x\| = 1$ in any C^* completion of $\vartheta(D_p)$ (Theorem 2.1.1 [32]).

Observe that (1.4) has the following useful symmetry. Let x_- be the generator of $\vartheta(D_{p^{-1}})$, (where we consciously abuse the notation as $p^{-1} \geq 1$), i.e.,

$$x_-^* x_- - p^{-1} x_- x_-^* = 1 - p^{-1}.$$

Then assignment $x \mapsto x_-^*$ can be extended to a $*$-algebra isomorphism

$$\kappa_p : \vartheta(D_p) \longrightarrow \vartheta(D_{p^{-1}}). \tag{1.6}$$

The coordinate algebra $\vartheta(D_p)$ can be completed to the C^*-algebra $C(D_p)$ with the norm

$$\|a\| = \sup_{\varrho} \|a\|_{\varrho}, \quad a \in \vartheta(D_p), \tag{1.7}$$

where the supremum is taken over all bounded representations $\varrho : \vartheta(D_p) \longrightarrow \mathbf{B}(\mathcal{H}_\varrho)$ of $\vartheta(D_p)$ and $\|\cdot\|_\varrho$ denotes the operator norm in the representation ϱ. The C^* algebra $C(D_p)$ is called a *universal enveloping algebra* of $\vartheta(D_p)$. Note that the norm (1.7) is well defined because $\|x\|_\varrho = 1$ for all ϱ.

Irreducible bounded representations of $\vartheta(D_p)$ are unitarily equivalent to one of the following representations.

1. For all $0 \leq \phi < 2\pi$, there is a one dimensional representation $\varrho_\phi : \vartheta(D_p) \longrightarrow \mathbb{C}$,

$$\varrho_\phi(x) = e^{i\phi}, \quad \varrho_\phi(x^*) = e^{-i\phi}. \tag{1.8}$$

2. There is also an infinitely dimensional representation $\varrho_\infty : \vartheta(D_p) \longrightarrow \mathbf{B}(\mathcal{H}_\infty)$, where $\mathcal{H}_\infty$ is generated by orthonormal vectors Ψ_n, $n = 0, 1, 2, \ldots$, and

$$\varrho_\infty(x)\Psi_n = \sqrt{1 - p^{n+1}}\Psi_{n+1}, \quad \varrho_\infty(x^*)\Psi_{n+1} = \sqrt{1 - p^{n+1}}\Psi_n, \quad \varrho_\infty(x^*)\Psi_0 = 0. \tag{1.9}$$

The infinite dimensional representation ϱ_∞ is faithful. Nonfaithful representations correspond to subsets of quantum discs. In particular, one-dimensional representations are also characters, that is, they correspond to the classical points of the quantum space. In the case of the quantum disc, one-dimensional representations describe the classical unit circle.

Let us adopt notational convention $x^{-n} \equiv (x^*)^n$. It follows from (1.4) that $(1 - xx^*)x = px(1 - xx^*)$, and, therefore,

$$(1 - xx^*)^n x^m = p^{mn} x^m (1 - xx^*)^n, \quad n \in \mathbb{N}, m \in \mathbb{Z}. \tag{1.10}$$

Moreover, it is easy to prove by induction that, for all $n \geq 0$,

$$(x^*)^n x^n = 1 + \sum_{m=1}^{n} (-1)^m p^{nm - \frac{m(m-1)}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_{p^{-1}} (1 - xx^*)^m, \tag{1.11a}$$

$$x^n (x^*)^n = 1 + \sum_{m=1}^{n} (-1)^m p^{-nm + \frac{m(m+1)}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_{p} (1 - xx^*)^m, \tag{1.11b}$$

where we used the standard p-deformed binomial coefficients

$$\begin{bmatrix} n \\ m \end{bmatrix}_{p} = \frac{[n]_p!}{[m]_p![n-m]_p!},$$
$$[n]_p! = [1]_p[2]_p \dots [n-1]_p[n]_p \text{ for } n \in \mathbb{N},\ [0]_p! = 1,$$
$$[n]_p = 1 + p + p^2 + \dots + p^{n-2} + p^{n-1} \text{ for } n \in \mathbb{N},\ [0]_p = 0. \tag{1.12}$$

Note that equation (1.11b) follows from (1.11a) by the application of the isomorphism (1.6).

It is now obvious that, for all $n, m \in \mathbb{Z}$,

$$x^n x^m = x^{n+m}(1 + Q^p_{n;m}(1 - xx^*)), \tag{1.13}$$

where $Q^p_{n;m}$ is a polynomial of degree at most $\min\{|m|, |n|\}$ and such that $Q^p_{n;m}(0) = 0$. For example, if $m \geq n \geq 0$, then $x^{-n}x^m = (x^{-n}x^n)x^{m-n}$, and then use (1.11a) and (1.10).

As elements $x^n x^m$, $n, m \in \mathbb{Z}$ obviously span $\vartheta(D_p)$ as a vector space, (1.13) means that also elements of the form

$$x^n(1 - xx^*)^m, \quad n \in \mathbb{Z},\ m \in \mathbb{N}_0, \tag{1.14}$$

span $\vartheta(D_p)$. In fact, using the infinite dimensional representation (1.9) one can prove that the above family forms a basis of $\vartheta(D_p)$. Indeed, suppose that

$$\sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} A_{nm} x^n (1 - xx^*)^m = 0,$$

for some coefficients $A_{mn} \in \mathbb{C}$, only a finite number of which are different from zero. Therefore, for any $k \geq 0$,

$$0 = \varrho_\infty\Big(\sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} A_{nm} x^n (1 - xx^*)^m\Big)\Psi_k = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} p^{mk} A_{nm} \varrho_\infty(x^n)\Psi_k$$
$$= \sum_{m=0}^{\infty} p^{mk}\Big(\sum_{n=1}^{k} A_{-nm}\sqrt{1-p^k}\dots\sqrt{1-p^{k-n+1}}\Psi_{k-n} + A_{0m}\Psi_k$$
$$+ \sum_{n=1}^{\infty} A_{nm}\sqrt{1-p^{k+1}}\dots\sqrt{1-p^{k+n}}\Psi_{k+n}.\Big)$$

As vectors Ψ_k are linearly independent and $p < 1$,

$$\sum_{m=0}^{\infty} p^{mk} A_{nm} = 0 \text{ for all } k \geq 0,\ -k \leq n,$$

i.e., for each $n \in \mathbb{Z}$, the polynomial $\sum_{m=0}^{\infty} A_{nm} t^m$ has an infinite number of distinct roots $p^k, k \geq 0$ if $n \geq 0$ or $k \geq -n$ otherwise. This is possible only if $A_{nm} = 0$ for all $n \in \mathbb{Z}, m \geq 0$. In addition, we have proven that the representation (1.9) is faithful.

1.2.2 The quantum torus

Quantum torus was defined in [36]. The coordinate algebra of the quantum torus $\vartheta(T_\phi)$, $\phi \in [0, 2\pi)$, is generated by unitary elements U, V which satisfy the following commutation relations

$$UV = e^{i\phi} VU,\ \ UV^* = e^{-i\phi} V^* U. \tag{1.15}$$

Obviously, the elements

$$V^n U^m, \quad n, m \in \mathbb{Z}, \tag{1.16}$$

form a basis of $\vartheta(T_\phi)$. The coordinate algebra $\vartheta(T_\phi)$ can be completed to the enveloping C^*-algebra $C(T_\phi)$ using representations of $\vartheta(T_\phi)$. The representation theory of $\vartheta(T_\phi)$ depends on whether ϕ is a rational or irrational multiple of 2π.

Suppose that $\phi = 2\pi\frac{M}{N}$, where $M, N \in \mathbb{N}$, $M < N$, and M and N are relatively prime. Then U^N and V^N are central in $\vartheta(T_\phi)$, and we can classify irreducible representations of $\vartheta(T_\phi)$ according to their eigenvalues. It turns out that irreducible representations of $\vartheta(T_\phi)$ include the ones isomorphic to one of the representations $\varrho_{\alpha\beta} : \vartheta(T_\phi) \to \mathbf{B}(\mathcal{H}^{\alpha\beta})$, $\alpha, \beta \in [0, 2\pi)$, where $\mathcal{H}^{\alpha\beta}$ is spanned by orthonormal vectors $\Psi_n^{\alpha\beta}$, $n \in \mathbb{Z}_N$, and

$$\varrho_{\alpha\beta}(U^{\pm 1})\Psi_n^{\alpha\beta} = e^{\pm i\frac{\alpha}{N}} e^{\pm 2\pi i n \frac{M}{N}} \Psi_n^{\alpha\beta}, \quad \varrho_{\alpha\beta}(V^{\pm 1})\Psi_n^{\alpha\beta} = e^{\pm i\frac{\beta}{N}} \Psi_{n\pm 1}^{\alpha\beta}. \tag{1.17}$$

On the other hand, if ϕ is an irrational multiple of 2π, then irreducible representations include the representations unitarily isomorphic to one of the $\varrho_\alpha : \vartheta(T_\phi) \to \mathbf{B}(\mathcal{H}^\alpha)$, $\alpha \in [0, 2\pi)$, where $\mathcal{H}^\alpha$ is the closure of the linear span of the family of orthonormal vectors Ψ_n^α, $n \in \mathbb{Z}$, and

$$\varrho_\alpha(U^{\pm 1})\Psi_n^\alpha = e^{\pm i\alpha} e^{\pm i n\phi} \Psi_n^\alpha, \quad \varrho_\alpha(V^{\pm 1})\Psi_n^\alpha = \Psi_{n\pm 1}^\alpha. \tag{1.18}$$

Note, that the quantum torus is not a type I C^*-algebra, therefore (cf. the discussion at the end of the chapter 3 in [1]) its irreducible representations cannot be explicitly listed.

Representations (1.18) are faithful. Representations (1.17) are not faithful. However, choose two sequences $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$, such that $\alpha_n, \beta_m \in [0, 2\pi)$, $m, n \in \mathbb{N}$ and $\alpha_n \neq \alpha_m, \beta_n \neq \beta_m$ if $n \neq m$. Then the representation

$$\varrho^{(\alpha_n)_{n\in\mathbb{N}}(\beta_n)_{n\in\mathbb{N}}} = \bigoplus_{m,n\in\mathbb{N}} \varrho^{\alpha_m\beta_n} : \vartheta(T_\phi) \longrightarrow \bigoplus_{m,n\in\mathbb{N}} \mathbf{B}(\mathcal{H}^{\alpha_m\beta_n}) \tag{1.19}$$

is faithful. Indeed, suppose that for some coefficients $A_{mn} \in \mathbb{C}$, $m, n \in \mathbb{Z}$, a finitely many of which are different from zero, $\varrho^{(\alpha_n)_{n\in\mathbb{N}}(\beta_n)_{n\in\mathbb{N}}}(\sum_{m,n\in\mathbb{Z}} A_{mn}V^mU^n) = 0$, i.e., for all $k, l \in \mathbb{Z}$ and $s \in \mathbb{Z}_N$,

$$0 = \varrho^{\alpha_k\beta_l}(\sum_{m,n\in\mathbb{Z}} A_{mn}V^mU^n)\Psi_s^{\alpha_k\beta_l} = \sum_{m,n\in\mathbb{Z}} A_{mn}e^{in\frac{\alpha_k}{N}}e^{2\pi is\frac{M}{N}}e^{im\frac{\beta_l}{N}}\Psi^{\alpha_k\beta_l}_{s+[m]_N}.$$

Then, for all $s \in \mathbb{Z}_N$, $m \in \{0, 1, 2, \ldots, N-1\}$,

$$\sum_{j,n\in\mathbb{Z}} A_{(m+jN)n}e^{in\frac{\alpha_k}{N}}e^{i\beta_l j} = 0 \text{ for all } k, l \in \mathbb{N}.$$

This, by the argument on the number of distinct roots of a finite polynomial, implies that $A_{mn} = 0$, for all $m, n \in \mathbb{Z}$.

1.3 Modules and comodules

1.3.1 Basic notation and terminology

Unless otherwise stated we work over a general commutative ring $\mathbb{K}$ with unit.

Categories of modules and comodules. Let C and H be coalgebras, and A and B algebras. We denote by ${}_A\mathcal{M}$, $\mathcal{M}_B$, ${}^H\mathcal{M}$, $\mathcal{M}^C$, ${}_A\mathcal{M}_B$, ${}^H\mathcal{M}^C$, ${}_A\mathcal{M}^C$, etc, respectively, the category of left A-modules, right B-modules, left H-comodules, right C-comodules, (A, B)-bimodules, (H, C)-bicomodules, left A-modules and right C-comodules such that the C-coaction commutes with the A-action, etc. Unadorned $\mathcal{M}$ will denote category of $\mathbb{K}$-modules. All algebraic objects considered belong to this category.

Tensor products. Let A be an algebra, and suppose that $M \in \mathcal{M}_A$, $N \in {}_A\mathcal{M}$. Then $M \otimes_A N$ will denote the tensor product over A. Unadorned tensor product denotes the tensor product over the ground ring $\mathbb{K}$. If A and B are algebras then, unless otherwise stated, we implicitly consider tensor product $A \otimes B$ as an algebra with multiplication $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ for all $a, a' \in A$, $b, b' \in B$. If A and B are C^*-algebras then $A\hat{\otimes}B$ denotes suitably completed tensor product over $\mathbb{C}$.

Multiplication and action Let A and B be algebras, let $M \in {}_A\mathcal{M}$, $N \in \mathcal{M}_B$. Occasionally we shall need to write explicitly as maps

the algebra multiplication	$m_A : A \otimes A \to A,$	$a \otimes a' \mapsto aa',$
the left A-action	${}_A\rho : A \otimes M \to M,$	$a \otimes m \mapsto am,$
the right B-action	$\rho_B : N \otimes B \to N,$	$n \otimes b \mapsto nb.$

We write m for m_A if the algebra is clear from the context.

Comultiplication, coaction and the Sweedler notation. Suppose C and H are coalgebras, $M \in {}^H\mathcal{M}$ and $N \in \mathcal{M}^C$. We denote the comultiplication by $\Delta : C \to C \otimes C$, the left H-coaction by ${}^H\rho : M \to H \otimes M$, the right C-coaction by $\rho^C : N \to N \otimes C$. Occasionally, to avoid confusion, we indicate the module being coacted on, writing ρ^C_N for $\rho^C : N \to N \otimes C$, and similarly for left coactions. We also use the Sweedler notation: $\Delta(c) = c_{(1)} \otimes c_{(2)}$, ${}^H\rho(m) = m_{(-1)} \otimes m_{(0)}$, $\rho^C(n) = n_{(0)} \otimes n_{(1)}$ for all $c \in C$, $m \in M$, $n \in N$, and the summation is implicitly understood.

Antipodes, counits and units. Let C be a coalgebra. We denote by $\varepsilon^C : C \to \mathbb{K}$ the counit of C. If there is no danger of confusion, we write ε for ε^C. If C is a Hopf algebra, then the antipode of C is denoted by $S_C : C \to C$. If A is an algebra, in most cases we use the symbol 1_A or simply 1 for the unit of A. Occasionally, we may need to write the unit explicitly as a map $\eta : \mathbb{K} \to A$, $k \mapsto k1_A$. Whenever it does not cause any ambiguity, we identify in notation the ground ring $\mathbb{K}$ with the subalgebra $\mathbb{K}1_A$ of A, i.e., depending on context, for any $k \in \mathbb{K}$, k may mean also $k1_A$ for some algebra A.

Double arrow notation Let $M, N \in \mathcal{M}$, and let $f, g : M \to N$ be morphisms in $\mathcal{M}$. The symbol

$$M \underset{g}{\overset{f}{\rightrightarrows}} N$$

denotes the morphism $f - g : M \to N$.

Coinvariants Suppose that C is a coalgebra and A is an algebra and a right (resp. left) C-comodule. Then the subalgebra of coinvariants of the right (resp. left) C-coaction is defined as

$$A^{\mathrm{co}C} = \{a \in A \mid \forall_{a' \in A}\ \rho^C(aa') = a\rho^C(a')\}$$
$$(\text{ resp. } {}^{\mathrm{co}C}A = \{a \in A \mid \forall_{a' \in A}\ {}^C\rho(a'a) = {}^C\rho(a')a\}\ ).$$

If C is a bialgebra and A is a *C-comodule algebra* (i.e. the C-coaction is an algebra map), then clearly $A^{\mathrm{co}C} = \{a \in A \mid \rho^C(a) = a \otimes 1_C\}$ (resp. ${}^{\mathrm{co}C}A = \{a \in A \mid {}^C\rho(a) = 1_C \otimes a\}$).

1.3.2 Noncommutative geometry and basic algebraic structures

The purpose of this section is to provide at least partial justification for geometric interpretations of certain basic algebraic structures. To simplify the reasoning we limit ourselves to compact metric spaces especially when dealing with tensor product. All examples of quantum spaces in this work are unital algebras corresponding to compact metric spaces.

To deal with geometric interpretation of tensor products we need the generalised *Stone-Weierstrass Theorem.*

Theorem 1.3.1. (Stone-Weierstrass) *(cf. [21]) Let X be a compact metric space. Suppose that B is a *-subalgebra of $C_0(X)$ such that $1 \in B$ and*

$$\forall_{\substack{x,y\in X\\ x\neq y}} \exists_{f\in B} f(x) \neq f(y) \tag{1.20}$$

Then B is dense in $C_0(X)$.

A subalgebra B of $C_0(X)$ which satisfies (1.20) is said to *separate* points of X. In particular, for any compact metric space X, $C_0(X)$ separates points of X.

Tensor products of algebras correspond to Cartesian products of quantum spaces. Indeed, let X, Y be compact metric spaces. View $C_0(X) \otimes C_0(Y)$ as a subalgebra of $C_0(X \times Y)$ by defining $(f \otimes g)((x,y)) = f(x)g(y)$, for all $f \in C_0(X)$, $g \in C_0(Y)$, $x \in X$, $y \in Y$. By the Stone-Weierstrass Theorem $C_0(X) \otimes C_0(Y)$ is dense in $C_0(X \times Y)$, i.e., $C_0(X \times Y) = \overline{C_0(X) \otimes C_0(Y)} = C_0(X)\hat{\otimes}C_0(Y)$.

Direct products of algebras are disjoint unions of quantum spaces. This is justified by the obvious equality $C_0(X \sqcup Y) = C_0(X) \oplus C_0(Y)$.

Quotients by (closed) ideals correspond to (closed) subsets. 'Closed' applies if the algebra considered is a C^*-algebra. Indeed, closed ideals J of $C_0(X)$ are in bijective correspondence with closed subsets $Y \subseteq X$ such that $f(y) = 0$, for all $y \in Y$. The kernel of the surjection $\pi : C_0(X) \to C_0(Y)$, $f \mapsto f|_Y$ is clearly equal to J. Notice that π is a surjective map because any map on a closed subset of compact metric space can be extended to the whole of the space.

(Closed) subalgebras are understood as quantum quotient spaces. Let $\pi : X \to Y$ be a continuous, surjective map of compact metric spaces. Then $C_0(Y) \simeq B$, where B is a subalgebra of $C_0(X)$ such that $f(x) = f(x')$ if $\pi(x) = \pi(x')$, for all $f \in B$, $x, x' \in X$.

Tensor products over subalgebras of the form $A \otimes_B A$, where $B \subseteq A$ is a subalgebra, are understood as quantum *fibre products.* Suppose that $\pi : X \to Y$ is a continuous, surjective map of compact metric spaces. *A fibre product*

$$X \times_\pi X = \{(x,x') \in X \times X \mid \pi(x) = \pi(x')\} \tag{1.21}$$

is a closed subset of $X \times X$. By the previous paragraph $C_0(Y)$ is a subalgebra of $C_0(X)$. By the definition, the tensor product $C_0(X) \otimes_{C_0(Y)} C_0(X)$ is isomorphic to $(C_0(X) \otimes C_0(X))/J$, where J is an ideal of $C_0(X) \otimes C_0(X)$ generated by elements of the form $fg \otimes h - f \otimes gh$, $f, h \in C_0(X)$, $g \in C_0(Y)$. The closure $\overline{J}$ of J consists of all functions $F \in C_0(X \times X)$, such that $F(z) = 0$, for all $z \in Z$, where Z is a closed subset of $X \times X$. As $J(X \times_\pi X) = \{0\}$, it follows that $X \times_\pi X \subseteq Z$. Suppose $Z \neq X \times_\pi X$, and $(x,x') \in Z \setminus (X \times_\pi X)$, i.e., $\pi(x) \neq \pi(x')$. As $C_0(Y)$ separates points of Y, there exists $g \in C_0(Y)$ such that $g(\pi(x)) \neq g(\pi(x'))$, Then $G = 1 \otimes g \circ$

$\pi - g \circ \pi \otimes 1 \in J$, and $G((x, x')) = g(\pi(x')) - g(\pi(x)) \neq 0$, i.e., $(x, x') \neq Z$. By contradiction, $Z = X \times_\pi X$. Consequently

$$C_0(X \times_\pi X) = C_0(X \times X)/\bar{J} = \overline{C_0(X) \otimes C_0(X)}/\bar{J}$$
$$= \overline{(C_0(X) \otimes C_0(X))/J} = \overline{C_0(X) \otimes_{C_0(Y)} C_0(X)}.$$

1.4 Projective modules

Suppose that B is a $\mathbb{K}$-algebra. A left (right) B-module P is called a *projective* module if, for any surjective map of left (resp. right) B-modules $\pi : M \to N$, and any left (resp. right) B module map $f : P \to N$, there exists a left (resp. right) B-module morphism $g : P \to M$ such that $\pi \circ g = f$. Projective left (resp. right) B-modules are flat as left (resp. right) B-modules.

Closely related to the notion of a projective module is the notion of a dual basis. Let I be an index set. To fix the notation assume that P is a left B-module. Denote by ${}^*P$ the set of left B-module morphisms $f : P \to B$. The family $e_i \in P$, $f_i \in {}^*P$, $i \in I$, is called a *dual basis* if for all $p \in P$, the set $\{i \in I \mid f_i(p) \neq 0\}$ is finite and $p = \sum_{i\in I} f_i(p)e_i$. A B-module P is projective if and only if it has a dual basis. P is finitely generated projective if and only if it has a finite dual basis.

Denote by $M_n(B)$ the set of $n \times n$ matrices with entries from B.

A B module is projective if and only if it is a direct summand of a free B-module. In particular suppose that P is a finitely generated projective left B-module. Then there exists $N \in \mathbb{N}$, an idempotent $E \in M_N(B)$ and injective B-morphism $\imath : P \to B^N$, such that $\imath(P) = B^N E$. Given a finite dual basis $e_i \in P$, $f_i \in {}^*P$, $i \in I = \{1, 2, \dots, N\}$ of a finitely generated projective B-module P, we define

$$\imath : P \to B^N,\ p \mapsto (f_i(p))_{i\in I},\ \ E = (f_j(e_i))_{i,j\in I} \in M_N(B). \tag{1.22}$$

Conversely, given an $n \in \mathbb{N}$ and an idempotent $E \in M_n(B)$, $B^n E$ is a finitely generated projective left B-module.

There exists a bijective correspondence between the isomorphism classes of finitely generated projective B modules and equivalence classes of idempotent matrices with entries in B. The idempotents $E \in M_n(B)$ and $E' \in M_m(B)$ are equivalent if for some $N \geq n, m$, there exists an invertible matrix $U \in M_N(B)$ such that $\imath^n_N(E) = U\imath^m_N(E')U^{-1}$, where for $n \leq m$, $\imath^n_m : M_n(B) \to M_m(B)$ enlarges matrix size by keeping the original contents in the upper left corner and filling the additional entries with zeroes.

Let C be a coalgebra, and suppose that $P \in {}_B\mathcal{M}^C$. P is called a *C-equivariantly projective* module if for every left B-module, right C-comodule epimorphism $\pi : M \to N$ split as a right C-comodule map, and every left B-module, right C-comodule morphism $f : P \to N$, there exists a left B-module, right C-comodule morphism $g : P \to M$, such that $\pi \circ g = f$, as illustrated with the following diagram:

$$\begin{tikzcd} & P \arrow[dl, dotted, "g"'] \arrow[d, "f"] & \\ M \arrow[r, shift left, "\pi"] & N \arrow[l, shift left, "\phi"] \arrow[r] & 0. \end{tikzcd} \tag{1.23}$$

The map $\phi : N \to M$ is a right C-comodule splitting morphism, i.e., $\pi \circ \phi = N$. It follows that π is a surjective (and ϕ injective) map. Note that if the ground ring $\mathbb{K}$ is a field, then the usual projectivity is a special case of C-equivariant projectivity, with a trivial coalgebra $C = \mathbb{K}$. Indeed, every surjective morphism is split as a map of vector spaces.

Lemma 1.4.1. *A module $P \in {}_B\mathcal{M}^C$ is C-equivariantly projective if and only if there exists a left B-module, right C-comodule splitting $\sigma : P \to B \otimes P$ of the left B-action, i.e., ${}_B\rho \circ \sigma = P$.*

Proof. Indeed, the left B-action ${}_B\rho : B \otimes P \to P$ is split as a right C-comodule with $\phi : P \to B \otimes P$, $p \mapsto 1_B \otimes P$, and hence, if P is C-equivariantly projective, the existence of splitting $\sigma : P \to B \otimes P$, follows from the following diagram:

$$\begin{tikzcd} & P \arrow[dl, dotted, "\sigma"'] \arrow[d, "P"] \\ B \otimes P \arrow[r, shift left, "{}_B\rho"] & P \arrow[l, shift left, "\phi"] . \end{tikzcd} \tag{1.24}$$

Conversely, let f, π, M, N, be as in (1.23), and suppose that there exists a left B-linear, right C-colinear splitting σ of the left B action. Then $g = {}_B\rho \circ (B \otimes \phi \circ f) \circ \sigma$ completes diagram (1.23) with desired properties. $\square$

Note that if $\mathbb{K}$ is a field, P is a $\mathbb{K}$-algebra, $B \subseteq P$ is a subalgebra, and there exists a unital left B-linear splitting $\sigma : P \to B \otimes P$ of a natural left B action on P (by algebra multiplication), then B is a direct summand in P as a left B-module. Indeed, canonical injection of subset into its overset $\iota : B \to P$ is split as a $\mathbb{K}$-linear map, with map, say, $f : P \to B$. Then

$$g = m_B \circ (B \otimes f) \circ \sigma : P \to B \tag{1.25}$$

is a left B-linear splitting of ι.

1.5 Entwining structures

Let C be a coalgebra and A an algebra. Entwining structures were introduced in [10] as a very general way of linking algebra structure on A and coalgebra structure on C.

If a $\mathbb{K}$-linear map $\psi : C \otimes A \to A \otimes C$ makes the following 'bow-tie' diagram commutative:

$$\begin{tikzcd}
 & C \otimes A \otimes A \arrow[ddl, "\psi\otimes A"'] \arrow[drr, "C\otimes m"] & & & & C \otimes C \otimes A \arrow[ddr, "C\otimes\psi"] & \\
 & & & C \otimes A \arrow[urr, "\Delta\otimes A"] \arrow[dr, "\varepsilon\otimes A"] \arrow[dd, "\psi"] & & & \\
A \otimes C \otimes A \arrow[ddr, "A\otimes\psi"'] & & C \arrow[ur, "C\otimes\eta"] \arrow[dr, "\eta\otimes C"'] & & A & & C \otimes A \otimes C \arrow[ddl, "\psi\otimes C"] \\
 & & & A \otimes C \arrow[ur, "A\otimes\varepsilon"'] \arrow[drr, "A\otimes\Delta"] & & & \\
 & A \otimes A \otimes C \arrow[urr, "m\otimes C"'] & & & & A \otimes C \otimes C &
\end{tikzcd} \tag{1.26}$$

then the triple $(A,C)_\psi$ is called *a right-right entwining structure* and ψ is called *a right-right entwining map*. Modifying in an obvious way the above diagram, we can define a *left-left entwining map* $\psi_L : A \otimes C \to C \otimes A$. Note, that if $\psi : C \otimes A \to A \otimes C$ is a bijective right-right entwining map, then, $\psi^{-1} : A \otimes C \to C \otimes A$ is a left-left entwining map. In what follows 'entwining map' and 'entwining structure' will mean the right-right version.

We use the following summation notation for an entwining map ψ, and, if entwining map is bijective, its inverse $\psi^{-1} : A \otimes C \to C \otimes A$:

$$\psi(c \otimes a) = a_\alpha \otimes c^\alpha,\ \psi^{-1}(a \otimes c) = c_A \otimes a^A, \text{ for all } c \in C,\ a \in A,$$

where, respectively, small Greek and large Latin letters are used for implicit summation indices.

Suppose that M is a right C-comodule with coaction ρ^C, and a right A-module. If, for all $m \in M$, $a \in A$,

$$\rho^C(ma) = m_{(0)}\psi(m_{(1)} \otimes a), \tag{1.27}$$

then M is called *an* $(A,C)_\psi$*-entwined module* or simply an entwined module. The category of $(A,C)_\psi$-entwined modules is denoted by $\mathcal{M}_A^C(\psi)$. The morphisms in $\mathcal{M}_A^C(\psi)$ are right A-module and right C-comodule maps. If ψ is a trivial entwining, i.e., for all $a \in A$, $c \in C$, $\psi(c \otimes a) = a \otimes c$, then $\mathcal{M}_A^C(\psi) = \mathcal{M}_A^C$.

Let $(A,C)_\psi$ be an entwining structure, and let P be an algebra and an entwined module. An algebra extension $B \subseteq P$ is called an em $(A,C)_\psi$-extension if and only if $B = P^{\mathrm{co}C}$. Of particular interests are $(A,C)_\psi$-extensions $B \subseteq A$. Such extensions are denoted by $A(B,C,\psi)$. In this case, if there exists a grouplike element $e \in C$ such that, for all $a \in A$, $\rho^C(a) = \psi(e \otimes a)$ then $A_e(B,C,\psi)$ is called an e-copointed $(A,C)_\psi$-extension.

Let A be an $(A,C)_\psi$ entwined module. Then

$$A^{\mathrm{co}C} = \{b \in A \mid \rho^C(b) = b\rho^C(1)\}. \tag{1.28}$$

Let C be a bialgebra, and suppose A is an algebra and a right C-comodule. Then A is a C-comodule algebra if and only if the map

$$\psi_{\text{can}} : C \otimes A \to A \otimes C, \; c \otimes a \mapsto a_{(0)} \otimes c a_{(1)} \tag{1.29}$$

is an entwining and A is an $(A, C)_{\psi_{\text{can}}}$-entwined module. If, in addition, C is a Hopf algebra with a bijective antipode, then ψ_{can} is also a bijective map with the inverse

$$\psi_{\text{can}}^{-1} : A \otimes C \to C \otimes A, \;\; a \otimes c \mapsto c S^{-1} a_{(1)} \otimes a_{(0)}. \tag{1.30}$$

Observe that if A is an $(A, C)_\psi$-entwined module, then the right C-coaction on A is uniquely determined by the entwining ψ and the value of the coaction on 1_A, with $\rho^C = 1_{(0)} \psi(1_{(1)} \otimes \cdot)$. Conversely, let $Y = \sum_i x_i \otimes c_i \in A \otimes C$ have the property

$$\sum_{ij} x_i x_{j\alpha} \otimes c_i{}^\alpha \otimes c_j = \sum_i x_i \otimes c_{i(1)} \otimes c_{i(2)},$$

then the map

$$\rho^C_{\psi, Y} : A \to A \otimes C, \;\; a \mapsto \sum_i x_i \psi(c_i \otimes a) \tag{1.31}$$

is a right C-coaction with $\rho^C_\psi(1) = \sum_i x_i \otimes c_i$.

Suppose that ψ is bijective. Then A is also a left C-comodule, with a left C-coaction defined as

$${}^C\rho_\psi : A \to C \otimes A, \;\; a \mapsto \psi^{-1}(a 1_{(0)} \otimes 1_{(1)}). \tag{1.32}$$

Observe, however, that A is not a C-bicomodule in general. If C is a Hopf algebra with a bijective antipode S, and A is a C-comodule algebra, then, by (1.30), the coaction (1.32) reads

$${}^C\rho_{\psi_{\text{can}}}(a) = S^{-1} a_{(1)} \otimes a_{(0)}, \text{ for all } a \in A. \tag{1.33}$$

Denote by $A^{\otimes^n} = A \otimes A \otimes \ldots \otimes A$ (n times). Then the map

$$\begin{aligned} \rho^C_{\otimes^n_\psi} &: A^{\otimes^n} \to A^{\otimes^n} \otimes C, \\ a_1 \otimes a_2 \otimes \ldots \otimes a_n &\mapsto 1_{(0)} a_{1\alpha_1} \otimes a_{2\alpha_2} \otimes \ldots \otimes a_{n\alpha_n} \otimes 1_{(1)}{}^{\alpha_1 \alpha_2 \ldots \alpha_n} \\ &= a_{1(0)} \otimes a_{2\alpha_2} \otimes \ldots \otimes a_{n\alpha_n} \otimes a_{1(1)}{}^{\alpha_2 \ldots \alpha_n} \end{aligned} \tag{1.34}$$

is a right C-coaction, making $A^{\otimes^n}$ an $(A, C)_\psi$-entwined module. In particular, if C is a bialgebra and A is a C-comodule algebra, then $\rho^C_{\otimes^n_{\psi_{\text{can}}}} = \rho^C_{\text{diag}}$, where

$$\begin{aligned} \rho^C_{\text{diag}} &: A^{\otimes^n} \to A^{\otimes^n} \otimes C, \\ a_1 \otimes a_2 \otimes \ldots \otimes a_n &\mapsto a_{1(0)} \otimes a_{2(0)} \otimes \ldots \otimes a_{n(0)} \otimes a_{1(1)} a_{2(1)} \cdots a_{n(1)} \end{aligned} \tag{1.35}$$

is the *diagonal C-coaction* on the tensor product of n copies of A.

1.6 Differential calculus

Let A be an algebra. An algebra $\Omega(A) = \bigoplus_{n=0}^{\infty} \Omega^n(A)$ equipped with a linear map $d : \Omega(A) \to \Omega(A)$ is called a *differential algebra* over A if

1. $\Omega^0(A) = A$, and $\Omega^n(A) \in {}_A\mathcal{M}_A$ for all $n \in \mathbb{N}_0$,
2. $\Omega^n(A)\Omega^m(A) \subseteq \Omega^{m+n}(A)$, i.e. $\Omega(A)$ is $\mathbb{N}_0$-graded,
3. $d(\Omega^n(A)) \subseteq \Omega^{n+1}(A)$,
4. $d \circ d = 0$ and for all $x \in \Omega^n(A), y \in \Omega(A)$, $d(xy) = d(x)y + (-1)^n x d(y)$ (the Leibniz rule).

Map d is referred to as a *derivation* or a *differential.* If, in addition, $\Omega(A)$ is generated as an algebra by A and $d(A)$, then it is called a *differential calculus.* Observe that when $\Omega(A)$ is a differential calculus, d is uniquely determined by its restriction to A. One refers to $\Omega^n(A)$, $n \in \mathbb{N}_0$, as the n-th order differential forms.

In what follows, we shall often omit the parenthesis around the argument of d, i.e., we shall write dx instead of $d(x)$.

An example of a differential algebra is an external algebra of differential forms on a C^∞-manifold.

A special case is the *universal differential calculus* ΩA. It is defined as follows:

1. $\Omega^1 A \equiv \ker m_A \subseteq A \otimes A$, where $m_A : A \otimes A \to A$ is the multiplication on A.
2. $\Omega^n A \equiv \Omega^1 A \otimes_A \Omega^1 A \otimes_A \ldots \otimes_A \Omega^1 A$ (n-times) for $n > 1$.
3. For all $x, y \in \Omega A$, $xy \equiv x \otimes_A y$.
4. For all $a \in A$, $da = 1 \otimes a - a \otimes 1$.

Suppose that C is a coalgebra and a differential algebra $\Omega(A)$ is a right C-comodule such that, for all $n \in \mathbb{N}_0$, $\Omega^n(A)$ is a C-subcomodule. If the differential d is a comodule morphism, then the differential algebra $\Omega(A)$ is said to be *right C-covariant.*

Suppose that A is an entwined module and $A \in \mathcal{M}_A^C(\psi)$. It is apparent that one can embed the n-th order universal differential forms $\Omega^n A$ in $A^{\otimes^{n+1}}$. One proves that the induced coaction $\rho^C_{\otimes_\psi^{n+1}} : A^{\otimes^{n+1}} \to A^{\otimes^{n+1}} \otimes C$ (1.34) restricts to the coaction on $\Omega^n A$. Hence, the universal differential calculus ΩA is right C-covariant.

Chapter 2

Quantum principal bundles

2.1 Introduction

Let X be a topological space with a continuous action $(x,g) \mapsto xg : X \times G \to X$ of a topological group G. X is called a *G-principal bundle* if the canonical map

$$X \times G \to X \times X, \quad (x,g) \mapsto (x,xg) \tag{2.1}$$

is proper and injective (a map $f : X \to Y$ between topological spaces X and Y is called proper if the inverse image of any compact subset in Y is a compact subset in X). X is a fibre bundle, its base space M is the space of orbits of G-action, and the fibre is homeomorphic to G. Observe that the image of the map (2.1) is the fibre product

$$X \times_\pi X = \{(x,x') \in X \times X | \pi(x) = \pi(x')\},$$

where $\pi : X \to M$ is the natural surjection onto the orbit space.

The most general, purely algebraic, direct generalisation of the above definition is as follows. We work over a commutative and unital ring $\mathbb{K}$. The space X corresponds to the algebra P (algebra of functions on the total space). The group G becomes, after dualisation, a coalgebra C which coacts on the right on P. Finally, the subalgebra $B = P^{\mathrm{co}C} \subseteq P$ of the coinvariants of the C-coaction is the algebra of functions on the orbit space of the G-action. The *canonical map*

$$\mathrm{can}_P^C : P \otimes_B P \to P \otimes C, \;\; p \otimes_B p' \mapsto pp'_{(0)} \otimes p'_{(1)} \tag{2.2}$$

is a dualisation of the corestriction to $X \times_\pi X$ of the map (2.1). The coaction of the coalgebra C is free if the map (2.2) is invertible. In this case, we call $P(B)^C$ *a C-coalgebra Galois extension.* Recall that $P \otimes_B P$ can be understood as the module of functions on the fibre product $X \times_\pi X$ (cf. page 9). In what follows we shall often omit the subscripts and superscripts from the symbol 'can' if the omission does not lead to an ambiguity.

If $\rho^C(1_P) = 1_P \otimes e$, for some group-like element $e \in C$, then $P(B)^C$ is called an *e-copointed or e-coaugmented C-coalgebra Galois extension* and we denote it by $P(B)^C_e$. If C is a Hopf algebra, and P is a C-comodule algebra, then $P(B)^C$ is called a *C-Hopf Galois extension.*

In what follows we shall also need a lifting of the canonical map to $P \otimes P$:

$$\widetilde{\mathrm{can}}_P^C : P \otimes P \to P \otimes C, \quad p \otimes p' \to pp'_{(0)} \otimes p'_{(1)}. \tag{2.3}$$

The map can_P^C is obviously left P-linear and right C-colinear, and so is its inverse if it exists. Therefore $(\mathrm{can}_P^C)^{-1}$ can be written as

$$(\mathrm{can}_P^C)^{-1}(p \otimes c) = p\tau_P^C(c), \text{ for all } p \otimes c \in P \otimes C, \tag{2.4}$$

where the map $\tau_P^C : C \to P \otimes_B P$ is called *a translation map*. We use an explicit 'Sweedler like' notation for the translation map,

$$\tau_P^C(c) = c^{[1]} \otimes_B c^{[2]}, \text{ for all } c \in C,$$

where an implicit summation is understood. The translation map has a number of useful properties (cf. 34.4 [13]). For all $c \in C$, $p \in P$,

$$c^{[1]}c^{[2]}{}_{(0)} \otimes c^{[2]}{}_{(1)} = 1_P \otimes c, \tag{2.5}$$

$$c_{(1)}{}^{[1]} \otimes_B c_{(1)}{}^{[2]} \otimes c_{(2)} = c^{[1]} \otimes_B c^{[2]}{}_{(0)} \otimes c^{[2]}{}_{(1)}, \tag{2.6}$$

$$c^{[1]}c^{[2]} = \varepsilon(c), \tag{2.7}$$

$$p_{(0)}p_{(1)}{}^{[1]} \otimes_B p_{(1)}{}^{[2]} = 1_P \otimes_B p, \tag{2.8}$$

$$c_{(1)}{}^{[1]} \otimes_B c_{(1)}{}^{[2]}c_{(2)}{}^{[1]} \otimes_B c_{(2)}{}^{[2]} = c^{[1]} \otimes_B 1_P \otimes_B c^{[2]}. \tag{2.9}$$

If $P(B)^C$ is a C-Hopf Galois extension, then also, for all $b \in B$, $c, c' \in C$,

$$c^{[1]}{}_{(0)} \otimes_B c^{[2]} \otimes c^{[1]}{}_{(1)} = c_{(2)}{}^{[1]} \otimes_B c_{(2)}{}^{[2]} \otimes Sc_{(1)}, \tag{2.10}$$

$$bc^{[1]} \otimes_B c^{[2]} = c^{[1]} \otimes_B c^{[2]}b, \tag{2.11}$$

$$(cc')^{[1]} \otimes_B (cc')^{[2]} = c^{[1]}c'^{[1]} \otimes_B c'^{[2]}c^{[2]}. \tag{2.12}$$

Entwinings and C-coalgebra Galois extensions are closely related. If $P(B)^C$ is a C-coalgebra Galois extension, then the map (cf. Theorem 2.7 [6])

$$\psi_{\mathrm{can}} : C \otimes P \to P \otimes C, \quad c \otimes a \mapsto \mathrm{can}_P^C(\tau_P^C(c)p) \tag{2.13}$$

is the unique entwining such that P is an entwined module. In particular, if $P(B)^C$ is a C-Hopf Galois extension, then $\psi_{\mathrm{can}}(c \otimes p) = p_{(0)} \otimes cp_{(1)}$ (cf. eq. (1.29)).

If $\psi_{\mathrm{can}} : C \otimes P \to P \otimes C$ is bijective, then the C-coalgebra Galois extension $P(B)^C$ is called a *symmetric C-coalgebra Galois extension*.

For algebras P which are left C-comodules, one can of course construct the left hand version of the above theory, in particular, there is the left canonical map

$$^C\mathrm{can}_P : P \otimes_B P \to C \otimes P, \quad a \otimes_B a' \mapsto {}^C\rho(a)a', \tag{2.14}$$

where $B = {}^{\mathrm{co}C}P = \{b \in P | \forall_{p \in P} {}^C\rho(pb) = {}^C\rho(p)b\}$ is a subalgebra of left coinvariants. If the map $^C\mathrm{can}_P$ is bijective, then we shall say that $^CP(B)$ is a C-coalgebra Galois extension.

Suppose that $P(B)^C$ is a symmetric C-coalgebra Galois extension. Then P is also a left C-comodule, with the coaction (1.32). One can prove that ${}^{coC}P = P^{coC} = B$, and

$$ {}^C\mathrm{can}_P = \psi^{-1} \circ \mathrm{can}_P^C. \tag{2.15}$$

Therefore ${}^C\mathrm{can}_P$ is also bijective, and ${}^CP(B)$ is a C-coalgebra Galois extension. Observe that, ${}^C\tau_P = ({}^C\mathrm{can}_P)^{-1}(\cdot \otimes 1_P) = (\mathrm{can}_P^C)^{-1} \circ \psi(\cdot \otimes 1_P) = \mathrm{can}_P^{C-1}(1_P \otimes \cdot) = \tau_P^C$.

Lemma 2.1.1. *[42] Let C be a coalgebra and P an algebra and a right C-comodule. Take any subalgebra $B \subseteq P^{coC}$ and define*

$$\mathrm{can}_{P(B)}^C : P \otimes_B P \to P \otimes C, \quad p \otimes_B p' \mapsto pp'_{(0)} \otimes p'_{(1)}. \tag{2.16}$$

If $\mathrm{can}_{P(B)}^C$ is bijective and P is a right faithfully flat B-module, then $B = P^{coC}$ and $P(B)^C$ is a C-coalgebra Galois extension. We will often refer to $\mathrm{can}_{P(B)}^C$ as a canonical map.

Proof. This lemma can be viewed as a special case of Theorem 4.12 [22], but it can also be proven directly as follows.

The canonical map $\mathrm{can}_{P(B)}^C$ maps $P \otimes_B P^{coC}$ onto $P\rho^C(1_P) \subseteq P \otimes C$. Moreover, $P\rho^C(1_P) \simeq P$, where the isomorphism and its inverse are given by:

$$p1_{(0)} \otimes 1_{(1)} \mapsto p1_{(0)}\varepsilon(1_{(1)}) = p, \quad p \mapsto p1_{(0)} \otimes 1_{(1)}.$$

On the other hand, there is an isomorphism $P \otimes_B B \simeq P$ given by the formulae $p \otimes_B b \mapsto pb$, $p \mapsto p \otimes_B 1_P$. This leads to the sequence of maps

$$P \simeq P \otimes_B B \xrightarrow{P\otimes\iota} P \otimes_B P^{coC} \simeq P\rho^C(1_P) \simeq P,$$

their composition is simply the identity map $p \mapsto p$. Hence $P \otimes \iota$ is a bijection and, by the faithful flatness of P as a right B-module, the map ι must also be bijective. □

Observe that the theory described in this chapter so far is purely algebraic and does not involve any topology. In fact it goes much farther then simply omitting a topology. Coalgebra C in the above context is usually understood as the algebra of functions on a 'quantum group', but we have not assumed any algebraic structure on C. In fact, in most considered cases, coalgebras involved are also algebras (and can be even completed to a C^*-algebra), but the problem is that in many naturally appearing examples, like some homogeneous spaces, the comultiplication or more frequently coaction does not agree with the multiplication in the coalgebra or the algebra on which it acts. Classically, the reason why functions on a group form a bialgebra and functions on a space on which a group acts form a comodule algebra is that the group multiplication and action, which dualise to the comultiplication and the coaction, are operations on points, and the function multiplication is also pointwise. But quantum spaces in general are nonlocal, they are not defined by points even if they contain some. Therefore it is reasonable to expect that many natural and important quantum spaces will be so nonlocal that they will not preserve nice relations between multiplications and 'co'-operations.

The word 'topological' in topological groups or topological G-spaces, i.e., the spaces on which some group G acts, means two things. First, it indicates that all of the spaces involved are topological. In the language of noncommutative geometry it corresponds to the requirement that the algebras considered are C^*-algebras. Secondly, it implies that group multiplications and actions are continuous. Dually it means that comultiplications and coactions, considered as maps between normed spaces are continuous. Suppose that A and C are C^*-algebras. Denote by $C\hat{\otimes}C$ and $A\hat{\otimes}C$ the appropriate completion of the respective algebraic tensor products. Suppose that C is a coalgebra with the comultiplication $\Delta : C \to C\hat{\otimes}C$, and A is is a right C-comodule with the coaction $\rho^C : A \to A\hat{\otimes}C$. Both maps may be continuous even if neither of them is algebraic. Therefore a coalgebra which is not an algebra may still be a topological quantum group and a coaction of such coalgebra — a topological coaction.

Another problem which requires explanation is in what sense we can regard C as a quantum group. A coalgebra C without an antipode can be understood as a set of functions only on a quantum semigroup. On the other hand, classically, if a semigroup acts freely on some topological space it must have a cancellation property, and it is well known fact in the topology, that a compact topological semigroup which has a cancellation property is a group. Therefore, it is reasonable to regard a coalgebra which coacts freely on some unital algebra as a set of functions on some 'quantum group'.

2.2 Strong connections

Definition 2.2.1. A *strong connection* on a C-coalgebra Galois extension $P(B)^C$ is a left P-linear projection $\Pi : \Omega^1 P \to \Omega^1 P$, such that $\ker \Pi = P\Omega^1 BP$, $(d - \Pi \circ d)(P) \subseteq \Omega^1 BP$, and the map $d - \Pi \circ d : P \to \Omega^1 P$ is right C-colinear, where C coacts on the second factor of $\Omega^1 BP \subseteq B \otimes P$.

In the classical geometry, a connection on a principal bundle X is a projection Π in the space of differential one-forms on X, which annihilates horizontal one-forms, that is one-forms which give zero when applied to vertical tangent vectors. Vertical tangent vectors are tangent vectors to the fibres of X, i.e., tangent to the curves of the form $xg(t)$, where $t \in \mathbb{R}$, $x \in X$ and $g : \mathbb{R} \to G$ is a curve in the structure group G of the bundle X. In other words, $I - \Pi$ selects differential forms which react to the changes along the base space ignoring 'gauge' changes along the fibre.

The existence of strong connections is equivalent to the existence of various other objects.

Proposition 2.2.2. *(Proposition 2.6 [8].) Let $P(B)^C$ be a C-coalgebra Galois extension. View $\Omega^1 BP \subseteq B \otimes P$ as a C-comodule via the coaction on the second factor. Then there exists a bijective correspondence between the following sets.*

1. *The set of strong connections on P.*

2. *The set of* strong covariant derivatives *on* P, *i.e., maps* $D : P \to \Omega^1 BP$, *such that* $D(1) = 0$ *and* $D(bp) = d(b)p + bD(p)$, *for all* $b \in B$, $p \in P$.

3. *Left B-linear, right C-colinear and unital splittings* $\sigma : P \to B \otimes P$ *of the multiplication* $m : B \otimes P \to P$.

This correspondence is given as follows. Given a strong covariant derivative D, *as in the point 2, define the strong connection* $\Pi_D : \Omega^1 P \to \Omega^1 P$, $pdq \mapsto pdq - pD(q)$. *Conversely, given a strong connection* Π, *define the strong covariant derivative* $D_\Pi = d - \Pi \circ d : P \to \Omega^1 BP$.

Similarly, the strong covariant derivative $D_\sigma : P \to \Omega^1 BP$, $p \mapsto 1 \otimes p - \sigma(p)$ *corresponds to a map* $\sigma : P \to B \otimes P$ *with the properties as in the point 3. Conversely, given a strong covariant derivative* D, *we define splitting* $\sigma_D : P \to B \otimes P$, $p \mapsto 1 \otimes p - D(p)$.

By the point 3 of the above proposition, the existence of a strong connection implies C-equivariant projectivity of P (Lemma 1.4.1). On the other hand, if the ground ring $\mathbb{K}$ is a field and $P(B)^C$ is an e-copointed C-coalgebra Galois extension, then C-equivariant projectivity of P implies existence of a strong connection (Proposition 2.9 [8]).

Definition 2.2.3. Suppose that $P(B)^C_e$ is an e-copointed symmetric C-coalgebra Galois extension. A map $\ell : C \to P \otimes P$ is called a *strong connection form* if it satisfies the following properties:

$$\ell(e) = 1_P \otimes 1_P, \tag{2.17}$$

$$\tilde{\mathrm{can}}^C_P \circ \ell(c) = 1 \otimes c, \tag{2.18}$$

$$({}^C\rho_{\psi_{\mathrm{can}}} \otimes P) \circ \ell(c) = c_{(1)} \otimes \ell(c_{(2)}), \tag{2.19}$$

$$(P \otimes \rho^C) \circ \ell(c) = \ell(c_{(1)}) \otimes c_{(2)}, \tag{2.20}$$

for all $c \in C$, where ${}^C\rho_{\psi_{\mathrm{can}}}$ is the left coaction defined in (1.32) and $\tilde{\mathrm{can}}^C_P$ is the lifting of the canonical map, defined in (2.3). We use a 'Sweedler-like' notation for a strong connection form:

$$\ell(c) = c^{\underline{[1]}} \otimes c^{\underline{[2]}}, \text{ for all } c \in C, \tag{2.21}$$

where an implicit summation is understood. Note that in the special case when C is a Hopf algebra with the bijective antipode, and $P(B)^C$ is a C-Hopf Galois extension, condition (2.19) reads (cf. (1.33))

$$c^{\underline{[1]}}{}_{(1)} \otimes c^{\underline{[1]}}{}_{(0)} \otimes c^{\underline{[2]}} = Sc_{(1)} \otimes c_{(2)}{}^{\underline{[1]}} \otimes c_{(2)}{}^{\underline{[2]}}, \text{ for all } c \in C. \tag{2.22}$$

Lemma 2.2.4. *(Lemma 2.3 [8].) Let* $P(B)^C_e$ *be a symmetric e-copointed C-coalgebra Galois extension. There exists a bijection between the set of strong connection forms* ℓ *on* P *and the set of left B-linear right C-colinear unital splittings* $\sigma : P \to B \otimes P$ *of left multiplication of* P *by* B. *This bijection and its inverse are explicitly given by the formulae:*

$$\sigma \mapsto (\ell : c \mapsto c^{[1]}\sigma(c^{[2]})), \quad \ell \mapsto (\sigma : p \mapsto p_{(0)}\ell(p_{(1)})). \tag{2.23}$$

Definition 2.2.5. A symmetric and e-copointed C-coalgebra Galois extension $P(B)^C_e$ is called a *principal coalgebra-Galois extension* if there exists a strong connection on P.

Theorem 2.2.6. *(Theorem 2.5 [7].) Let $P(B)^C_e$ be a principal C-coalgebra Galois extension over a field $\mathbb{K}$. Then P is a faithfully flat left and right B-module.*

Recall that a coalgebra C is said to be *coseparable* if there exists a (C, C)-bicolinear retraction of the comultiplication, i.e., a map $\delta : C \otimes C \to C$ such that $\delta \circ \Delta = C$. If C is a coseparable coalgebra, then C-coalgebra Galois extensions have the following property.

Theorem 2.2.7. *([4], Theorem 4.6) Let $\mathbb{K}$ be a field and $P_e(B, C, \psi)$ be an e-copointed $(P, C)_\psi$-extension with ψ bijective. If C is a coseparable coalgebra and the lifted canonical map $\tilde{\mathrm{can}}^C_P : P \otimes P \to P \otimes C$ is surjective, then the canonical map $\mathrm{can}^C_P : P \otimes_B P \to P \otimes C$ is bijective and $P(B)^C_e$ is a principal C-coalgebra Galois extension.*

Proposition 2.2.8. *Let $\mathbb{K}$ be a field. Suppose that H, K are Hopf algebras and that $f : H \to K$ is a surjective Hopf algebra morphism. Let P be an H-comodule algebra. Obviously P and H are right K-comodule algebras with the right K-coactions*

$$\rho^K_P : P \to P \otimes K, \ \ p \mapsto p_{(0)} \otimes f(p_{(1)}),$$
$$\rho^K_H : H \to H \otimes K, \ \ h \mapsto h_{(1)} \otimes f(h_{(2)}),$$

and similarily H is a left K-comodule algebra with the left K-coaction

$${}^K\rho_H : H \to K \otimes H, \ \ h \mapsto f(h_{(1)}) \otimes h_{(2)}.$$

Let B=P^{coH} and $A = P^{coK}$. Suppose that $H^{coK} = {}^{coK}H$. Then the following conditions are satisfied.

1. *H^{coK} is a Hopf subalgebra of H.*
2. *A is an H^{coK} comodule subalgebra of P.*
3. *$B = A^{co(H^{coK})}$.*
4. *If $P(B)^H$ is an H-Hopf Galois extension, and P is flat as a left and right B-module, then $A(B)^{H^{coK}}$ is an H^{coK}-Hopf Galois extension.*
5. *If $P(B)^H$ is a principal extension, then $A(B)^{H^{coK}}$ is a principal extension.*

Proof. **1)** Let $h \in H^{coK} = {}^{coK}H$. Then

$$(H \otimes \rho^K_H) \circ \Delta(h) = h_{(1)(1)} \otimes h_{(1)(2)} \otimes f(h_{(2)}) = \Delta(h) \otimes 1,$$

and similarily,

$$({}^K\rho_H \otimes H) \circ \Delta(h) = f(h_{(1)}) \otimes h_{(2)(1)} \otimes h_{(2)(2)} = 1 \otimes \Delta(h).$$

Hence $\Delta(H^{\mathrm{co}K}) \subseteq {}^{\mathrm{co}K}H \otimes H^{\mathrm{co}K}$. Furthermore,

$$\rho_H^K \circ S_H(h) = S_H(h)_{(1)} \otimes f(S_H(h)_{(2)}) = S_H h_{(2)} \otimes S_K(f(h_{(1)})) = S_H h \otimes S_K(1).$$

2) Let $a \in A$. Then

$$(A \otimes \rho_H^K) \circ \rho^H(a) = a_{(0)(0)} \otimes a_{(0)(1)} \otimes f(a_{(1)}) = \rho^H(a) \otimes 1,$$

hence $\rho^H(A) \subseteq P \otimes H^{\mathrm{co}K}$, and then

$$(\rho_P^K \otimes H^{\mathrm{co}K}) \circ \rho^H(a) = a_{(0)} \otimes f(a_{(1)(1)}) \otimes a_{(1)(2)} = a_{(0)} \otimes 1 \otimes a_{(1)},$$

hence $\rho^H(A) \subseteq A \otimes H^{\mathrm{co}K}$.
3) Obviously, $B \subseteq A^{\mathrm{co}(H^{\mathrm{co}K})} \subseteq A$. On the other hand, by the previous point, $\rho^{H^{\mathrm{co}K}}(A) = \rho^H(A) \subseteq A \otimes H^{\mathrm{co}K}$, therefore, $A^{\mathrm{co}(H^{\mathrm{co}K})} = A^{\mathrm{co}H} = A \cap B = B$.
4) Suppose that $h \in H^{\mathrm{co}K}$. Then

$$(P \otimes_B \rho_P^K) \circ \tau_P^H(h) = \tau_P^H(h_{(1)}) \otimes f(h_{(2)}) = \tau_P^H(h) \otimes 1,$$

and

$$\begin{aligned}(T_{P,H} \circ \rho_P^K \otimes_B P) \circ \tau_P^H(h) &= f(h^{[1]}{}_{(1)}) \otimes h^{[1]}{}_{(0)} \otimes_B h^{[2]} = f(S_H h_{(1)}) \otimes h_{(2)}{}^{[1]} \otimes_B h_{(2)}{}^{[2]} \\ &= S_K f(h_{(1)}) \otimes h_{(2)}{}^{[1]} \otimes_B h_{(2)}{}^{[2]} = 1 \otimes \tau_P^H(h),\end{aligned}$$

where $T_{P,H} : P \otimes H \to H \otimes P$, $p \otimes h \mapsto h \otimes p$ is the flip map. Therefore, by the flatness of P as a right and left B-module, $\tau_P^H(H^{\mathrm{co}K}) \subseteq A \otimes_B A$.
5) If $\ell_P : H \to P \otimes P$ is a strong connection form on P, then $\ell_A = \ell_P|_{H^{\mathrm{co}K}} : H^{\mathrm{co}K} \to A \otimes A$ is a strong connection form on A. The proof that ℓ_A is well-defined and has all the required properties is analogues to the proof in the previous point. □

2.3 Quantum vector bundles and associated bundles

2.3.1 Vector bundles

Suppose that $\mathbb{K}=\mathbb{C}$. Let M and X be locally compact Hausdorff topological spaces, and let V be a finitely dimensional vector space. Let $\pi : X \to M$ be a surjective continuous map. The bundle (X, M, π) is called a *(locally trivial) vector bundle* if there exists a family of open sets $U_\alpha \subseteq M$, $\alpha \in I$ covering M, i.e., $\bigcup_\alpha U_\alpha = M$, and such that, for all $\alpha \in I$, there exists a homeomorphism

$$\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times V, \tag{2.24}$$

such that, for each $\alpha, \beta \in I$, there exists a continuous map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V)$, where $GL(V)$ denotes the group of linear invertible transformations of V, such that the transition map has the form

$$\phi_\alpha \circ \phi_\beta^{-1}\Big|_{(U_\alpha \cap U_\beta)\times V} : (U_\alpha \cap U_\beta) \times V \to (U_\alpha \cap U_\beta) \times V,\ (m, v) \mapsto (m, v g_{\alpha\beta}(m)), \tag{2.25}$$

where we assumed that $GL(V)$ acts on V on the right.

A *continuous section* of a vector bundle is a map $s : M \to X$, such that, for all $m \in M$, $\pi \circ s(m) = m$. We denote by $\Gamma_0(X)$ the set of continuous sections of X. Note that, for all $m \in M$, $\pi^{-1}(m)$ has the natural structure of a vector space. Indeed, let $x, y \in \pi^{-1}(m)$, $a, b \in \mathbb{C}$, and suppose that $m \in U_\alpha$. Then we define $ax + bx' = \phi_\alpha^{-1}((m, av + bv'))$, where $(m, v) = \phi_\alpha(x)$, $(m, v') = \phi_\alpha(x')$. It is clear that this definition does not depend on the choice of α. Using this, $\Gamma_0(X)$ has a natural structure of a left $C_0(M)$-module with pointwise multiplication, i.e., for all $m \in M$, $f \in C_0(M)$ and $s \in \Gamma_0(X)$, $(fs)(m) = f(m)s(m)$.

In fact, it turns out that, if M is compact, $\Gamma_0(X)$ is a finitely generated projective $C(M)$-module. Conversely, by the *Serre-Swan Theorem*, any finitely generated projective module of the algebra $C(M)$ of continuous functions on a compact space M, corresponds to a locally trivial vector bundle with the base space M.

Turning to noncommutative geometry, for a commutative ground ring $\mathbb{K}$ and unital $\mathbb{K}$-algebra A, we define *a quantum vector bundle* over the quantum space A as a finitely generated projective A-module.

2.3.2 Associated vector bundles

Suppose that G is a topological group and X is a G-principal bundle with the base space M, where G acts on X on the right. Let $\varrho : G \to GL(V)$ be a representation of G as linear operators acting on the right on a finitely dimensional $\mathbb{C}$-vector space V. Define the right G-action on $X \times V$ by

$$(X \times V) \times G \ni ((x, v), g) \mapsto (xg, v\varrho(g)) \in X \times V. \tag{2.26}$$

Then the space of orbits $(X \times V)/G$ is called *an associated vector bundle* and it can be shown that, under certain topological conditions, it is a locally trivial vector bundle with a base space M.

Note that continuous sections $s \in \Gamma_0((X \times V)/G)$ are in bijective correspondence with continuous functions $f \in C_\varrho(X, V)$, where

$$C_\varrho(X, V) = \{f : X \to V \mid \forall_{x \in X, g \in G},\ f(xg) = f(x)\varrho(g)\}. \tag{2.27}$$

Turning to noncommutative geometry, we can dualise the notion of $C_\varrho(X, V)$ to obtain the following definition. In what follows in this subsection, we assume that the ground ring $\mathbb{K}$ is a field.

Definition 2.3.1. Suppose that $P(B)^C$ is a C-coalgebra Galois extension. Let V be a finite dimensional vector space and a right C-comodule. Then we call $\mathrm{Hom}^C(V, P)$ a *quantum associated bundle* .

The following theorem states that, under certain conditions, quantum associated vector bundles are quantum vector bundles, i.e., finitely generated projective modules.

Theorem 2.3.2. *(Theorem 2.13 [8].) Suppose that $P(B)_e^C$ is a principal C-coalgebra Galois extension and V is a finitely dimensional vector space and a C-comodule. Let $(e_i)_{i=1,2,\dots,n}$*

be a basis of V, and suppose that $\rho^C(e_i) = \sum_{k=1}^n e_k \otimes e_{ki}$, where $e_{ij} \in C$, $i, j = 1,2,\ldots,n$. Choose some dual basis $p_\mu \in P$, $p^\mu : P \to \mathbb{K}$, $\mu \in I$ of the $\mathbb{K}$-vector space P. Take any left B-module morphism $g : P \to B$, such that $g(1) = 1$, which exists by Proposition 2.2.2 and (1.25). Let $\ell : C \to P \otimes P$, $c \mapsto c^{\underline{[1]}} \otimes c^{\underline{[2]}}$ be a strong connection form on P. Then $\mathrm{Hom}^C(V,P)$ is a finitely generated left B-module. In particular, it has a dual basis, which explicitly reads:

$$\begin{aligned} &f_{\mu k} \in \mathrm{Hom}^C(V,P),\ e_i \mapsto p^\mu(e_{ki}{}^{\underline{[1]}})e_{ki}{}^{\underline{[2]}},\\ &f^{\mu k} : \mathrm{Hom}^C(V,P) \to B, f \mapsto g(f(e_k)p_\mu),\\ &\mu \in J,\ k = 1,2,\ldots,n,\\ &\text{where } J = \{\nu \in I \mid \exists_{1\le k,i\le n} p^\nu(e_{ki}{}^{\underline{[1]}})e_{ki}{}^{\underline{[2]}} \neq 0\}. \end{aligned} \tag{2.28}$$

In particular by (1.22) this yields an idempotent

$$E_{(\mu i)(\nu j)} = f^{\nu j}(f_{\mu i}) = g(p^\mu(e_{ij}{}^{\underline{[1]}})e_{ij}{}^{\underline{[2]}}p_\nu). \tag{2.29}$$

Let us apply the above theorem to the special case of associated line bundles, i.e., to the situation when V is one dimensional. Then, for any nonzero $v \in V$, $\rho^C(v) = v \otimes u$, where $u \in C$ is necessarily group-like. Write value of the strong connection form at u explicitly as

$$\ell(u) = \sum_{i=1}^m l(u)_i \otimes r(u)_i, \tag{2.30}$$

where $\{l(u)_i, 1 \le i \le m\}$ and $\{r(u)_i, 1 \le i \le m\}$ are (separately) linearly independent sets of vectors in P. In particular, we can extend $\{l(u)_i, 1 \le i \le m\}$ to the basis of P. If we use this basis, the formula for the idempotent (2.29) becomes $E_{ij} = g(r(u)_i l(u)_j)$, $1 \le i,j \le m$.

We will prove that $r(u)_i l(u)_j \in B$ for all $1 \le i,j \le m$. Indeed, as u is group-like, the properties (2.19) and (2.20), and the linear independence of $\{l(u)_i, 1 \le i \le m\}$ and $\{r(u)_i, 1 \le i \le m\}$ yield

$$\psi_{\mathrm{can}}^{-1}(l(u)_i \otimes e) = u \otimes l(u)_i,\ \ \rho^C(r(u)_i) = r(u)_i \otimes u,\ \ \text{for all } i = 1,2,\ldots,m. \tag{2.31}$$

Therefore, for all $i,j = 1,2,\ldots,m$,

$$\begin{aligned} \rho^C(r(u)_i l(u)_j) &= r(u)_{i(0)}\psi_{\mathrm{can}}(r(u)_{i(1)} \otimes l(u)_j)\\ &= r(u)_i\psi_{\mathrm{can}}(u \otimes l(u)_j) = r(u)_i l(u)_j \otimes e. \end{aligned}$$

Hence we have proven that, in the case of associated line bundles, an idempotent (2.29) has the form

$$E_{ij} = r(u)_i l(u)_j,\ \ 1 \le i,j \le m. \tag{2.32}$$

Let us consider Theorem 2.3.2 in yet another situation.

Corollary 2.3.3. *(to Proposition 2.2.8.) Suppose that C is a Hopf algebra, and $P(B)^C$ is a principal Hopf Galois extension. Suppose further that there exists a surjective morphism of Hopf algebras $f : C \to K$, which makes C a (K,K)-bicomodule and P a right K-comodule algebra (cf. Proposition 2.2.8). Assume that the antipode in K is bijective and that $C^{coK} = {}^{coK}C$, i.e., all the assumptions of Proposition 2.2.8 are satisfied. Furthermore, suppose that the C-coaction $\rho^C : V \to V \otimes C$ on the finite dimensional vector space V is such that $\rho^C(V) \subseteq V \otimes C^{coK}$, i.e., $e_{ij} \in C^{coC}$, $i,j = 1,2,\ldots,n$ (cf. Proposition 2.2.8). Then the formulae (2.28) and (2.29) define, respectively, a dual basis and a projector for a quantum vector bundle associated to the principal extension $P^{coC}(B)^{C^{coK}}$.*

2.4 Example: A noncommutative Hopf fibration

2.4.1 Introduction

View the classical three-sphere as

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}. \tag{2.33}$$

There is a free action of the group $U(1)$ on S^3:

$$S^3 \times U(1) \to S^3,\ ((z_1, z_2), e^{i\phi}) \mapsto (z_1 e^{i\phi}, z_2 e^{i\phi}). \tag{2.34}$$

It is easy to see that $S^3/U(1) \simeq S^2$, where the two sphere

$$S^2 = \{(r, z) \in \mathbb{R}_0^+ \times \mathbb{C} \mid r^2 + |z|^2 = 1\}, \tag{2.35}$$

and thus S^3 is a principal bundle over the two-sphere S^2, with the fibre $U(1)$. This fibration of S^3 is called the *Hopf fibration*.

Furthermore, there exists a right $\mathbb{Z}_N$-action on S^3 given by

$$\rho^N_{m,n} : S^3 \times \mathbb{Z}_N \to S^3,\ ((z_1, z_2), k_N) \mapsto (z_1 e^{2\pi i \frac{km}{N}}, z_2 e^{2\pi i \frac{kn}{N}}), \tag{2.36}$$

where $1 \leq m, n < N$ and m, n are relatively prime with N. The quotient space of this action $L^N_{m,n} = S^3/(\rho^N_{m,n})$ is called the *lens space*. Clearly, there exists a free action of $U(1)$ on $L^N_{m,n}$,

$$L^N_{m,n} \times U(1) \to L^N_{m,n},\ ([(z_1, z_2)], e^{i\phi}) \mapsto [(z_1 e^{i\phi/N}, z_2 e^{i\phi/N})],\ 0 \leq \phi < 2\pi,$$

and $L^N_{m,n}$ is a principal bundle over S^2.

The *Dirac monopole* is a solution to Maxwell's equations, which describes a point source of a magnetic field. The vector potential of this magnetic field is not well defined on the whole of the space. It was however recognised ([43]) that this vector potential can be reinterpreted as a connection form on a nontrivial principal $U(1)$-bundle over the two-sphere S^2. The quantised magnetic charge $n = 1$ corresponds to the connection on the Hopf fibration. The magnetic charges of all the other (integer) values n correspond to connections on the lens spaces $L_n = L^n_{1,1}$.

The results in the remaining part of this section were published as [5]. We work over complex numbers, i.e., $\mathbb{K} = \mathbb{C}$.

2.4.2 A quantum Hopf fibration

The aim of this section is to show that the quantum contact 3-sphere defined in [33] is a Hopf-Galois extension of its algebra of coinvariants. The latter is the algebra of functions of the quantum 2-sphere also defined in [33].

The polynomial $*$-algebra $A(S^3_\mu)$ that underlies the Omori-Maeda-Miyazaki-Yoshioka quantum contact 3-sphere is generated by a selfadjoint μ and by a, b, a^*, b^* with relations

$$ba = ab, \qquad ab^* = (1-\mu)b^*a, \tag{2.37a}$$
$$\mu a - a\mu = \mu a \mu, \quad \mu b - b\mu = \mu b \mu, \tag{2.37b}$$
$$aa^* - (1-\mu)a^*a = \mu, \quad bb^* - (1-\mu)b^*b = \mu, \tag{2.37c}$$
$$a^*a + b^*b = 1. \tag{2.37d}$$

From the above relations it follows that

$$aa^* + bb^* = 1 + \mu. \tag{2.38}$$

Note also that (2.37b) is equivalent to

$$\mu a(1+k\mu) = (1+(k+1)\mu)a\mu, \qquad \mu b(1+k\mu) = (1+(k+1)\mu)b\mu,$$

for all $k \in \mathbb{Z}$. We denote by $A'(S^3_\mu)$ the polynomial $*$-algebra $A(S^3_\mu)$ with the generator μ required to be invertible (i.e. with adjoined μ^{-1}).

It turns out convenient for our purposes to employ a certain *μ-regulated smooth algebra* $A^\infty(S^3_\mu)$, which is defined and studied in [33]. The algebra $A^\infty(S^3_\mu)$, called *noncommutative contact algebra* on S^3, contains densely the polynomial $*$-algebra $A'(S^3_\mu)$. Also, it contains $f(\mu)$ for any formal power series f, and the following relations are fulfilled

$$af(\mu) = f(\frac{\mu}{1+\mu})a, \qquad f(\mu)a = af(\frac{\mu}{1-\mu}), \tag{2.39a}$$
$$bf(\mu) = f(\frac{\mu}{1+\mu})b, \qquad f(\mu)b = bf(\frac{\mu}{1-\mu}). \tag{2.39b}$$

In particular, for all $k \in \mathbb{Z}$, the elements $1+k\mu$ are invertible and have square root in $A^\infty(S^3_\mu)$. In the sequel we shall need their inverses as well as the square roots, which satisfy the following relations, for all $k \in \mathbb{Z}$,

$$a\mu(1+k\mu)^{-1} = \mu(1+(k+1)\mu)^{-1}a, \qquad b\mu(1+k\mu)^{-1} = \mu(1+(k+1)\mu)^{-1}b \tag{2.40}$$

and

$$a\sqrt{1+k\mu} = \frac{\sqrt{1+(k+1)\mu}}{\sqrt{1+\mu}}a, \qquad b\sqrt{1+k\mu} = \frac{\sqrt{1+(k+1)\mu}}{\sqrt{1+\mu}}b. \tag{2.41}$$

Although μ is a generator it can be regarded as a noncentral formal parameter, cf. [33] for a precise meaning of this statement. Also, from the defining relations of

(2.37) it is apparent that $A^\infty(S^3_\mu)$ is a $\mathbb{Z}$-graded algebra with the grading defined by setting

$$\deg(a) = \deg(b) = 1,\ \deg(a^*) = \deg(b^*) = -1,\ \deg(\mu) = 0.$$

This in turn allows us to view $A^\infty(S^3_\mu)$ as a comodule algebra of the Hopf algebra H of functions on $U(1)$. Explicitly, $H = \mathbb{C}[u, u^{-1}]$ is the algebra of Laurent polynomials in one variable u (i.e., u^{-1} is the multiplicative inverse of u), and it is a $*$-algebra with $u^* = u^{-1}$. A Hopf algebra structure of H is determined by $\Delta(u) = u \otimes u$, $\varepsilon(u) = 1$ and $S(u) = u^{-1}$. The grading of $A^\infty(S^3_\mu)$ makes it a right comodule algebra with the coaction given on homogeneous elements by

$$\rho^H(x) = x \otimes u^{\deg(x)}.$$

Thus explicitly on generators the coaction comes out as

$$\rho^H(a) = a \otimes u,\ \ \rho^H(b) = b \otimes u,$$
$$\rho^H(a^*) = a^* \otimes u^{-1},\ \ \rho^H(b^*) = b^* \otimes u^{-1},\ \ \rho^H(\mu) = \mu \otimes 1.$$

Note that the coaction ρ^H is compatible with the $*$-structure (it is a $*$-algebra homomorphism).

The definition of the coaction in terms of the grading immediately implies that the coinvariant subalgebra coincides with the zero-degree subalgebra, i.e.,

$$A^\infty(S^2_\mu) := \{x \in A^\infty(S^3_\mu) \mid \rho^H(x) = x \otimes 1\} = \{x \in A^\infty(S^3_\mu) \mid \deg(x) = 0\}.$$

Using equation (2.40) it is immediate to verify that μ is a central element in $A^\infty(S^2_\mu)$. It can be also seen that $A^\infty(S^2_\mu)$ is the commutant of μ in $A^\infty(S^3_\mu)$.

The relations (2.37), (2.40) and (2.41) provide us with a deeper insight into the structure of $A^\infty(S^2_\mu)$. With their help we can establish that $A^\infty(S^2_\mu)$ contains a (dense) polynomial $*$-algebra $A'(S^2_\mu)$ generated by

$$X = X^* = aa^* - (\mu + 1)/2, \quad Z = ab^*, \quad Z^* = ba^*,$$

and self-adjoint (invertible) element μ. Notice that $A'(S^2_\mu)$ is contained strictly in the commutant of μ in $A'(S^3_\mu)$, which coincides with the grade zero subalgebra of $A'(S^3_\mu)$ and also with the H-coinvariant subalgebra of $A'(S^3_\mu)$. Similar observations hold for $A(S^2_\mu)$, defined as the $*$-algebra obtained from $A'(S^2_\mu)$ by omitting the invertibility of μ.

The relations in $A^\infty(S^2_\mu)$ are derived from the relations in $A^\infty(S^3_\mu)$ and come out as

$$\mu X - X\mu = 0,\ \mu Z - Z\mu = 0, \tag{2.42a}$$
$$XZ - ZX = -\mu Z, \tag{2.42b}$$
$$ZZ^* - Z^*Z = -2\mu X, \tag{2.42c}$$
$$\left(X + \frac{\mu}{2}\right)^2 + ZZ^* = \frac{1}{4}, \tag{2.42d}$$

Note that from the relations above it follows that there is also a second radial relation

$$(X - \frac{\mu}{2})^2 + Z^*Z = \frac{1}{4}.$$

Since μ is central, it could be possible to consider μ as a formal parameter and specify it to a numerical value, we shall comment on this issue in the final part of this section.

We claim that in a 'dual' sense the quantum contact three sphere S^3_μ is a total space of a quantum $U(1)$-principal bundle over the quantum two-sphere S^2_μ, i.e., $A^\infty(S^3_\mu)(A^\infty(S^2_\mu))^H$ is an H-Hopf Galois extension. This claim is proven by explicit construction of the inverse to the canonical map can^H_P.

To relieve the notation we write P for $A^\infty(S^3_\mu)$, B for $A^\infty(S^2_\mu)$ and can for can^H_P. Consider the map $\mathrm{can}^{-1} : P \otimes H \to P \otimes_B P$ defined, for all $x \in P$ and $n \in \mathbb{N}$, by

$$\mathrm{can}^{-1}(x \otimes u^n) = \sum_{k=0}^{n} \binom{n}{k} x(a^*)^{n-k}(b^*)^k \otimes_B b^k a^{n-k}, \tag{2.43a}$$

$$\mathrm{can}^{-1}(x \otimes u^{-n}) = x(1+n\mu)^{-1} \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \otimes_B (b^*)^k (a^*)^{n-k}, \tag{2.43b}$$

where $\binom{n}{k}$ are the usual binomial coefficients. Directly from the definition it follows that can^{-1} is a left P-module map. Furthermore, the degree counting on the right hand side and the comparison of the powers of u immediately confirm that can^{-1} is a right H-comodule map. Before we prove that can^{-1} is the inverse map to can we note that, for all $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} \binom{n}{k} (a^*)^{n-k}(b^*)^k b^k a^{n-k} = 1, \tag{2.44a}$$

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k (b^*)^k (a^*)^{n-k} = 1 + n\mu. \tag{2.44b}$$

The formulae (2.44) are most easily proven by induction. They are clearly satisfied for $n = 1$. Next, assume that they hold for $n-1$ with $n \geq 2$. Using equations (2.37d), (2.38) and the well-known formula

$$\sum_{l=0}^{k} (-1)^l \binom{n}{k-l} = \binom{n-1}{k},$$

observe that

$$\sum_{k=0}^{n} \binom{n}{k} (a^*)^{n-k}(b^*)^k b^k a^{n-k} = \sum_{k=0}^{n-1} \binom{n-1}{k} (a^*)^{n-1-k}(b^*)^k b^k a^{n-1-k}, \tag{2.45a}$$

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k (b^*)^k (a^*)^{n-k} = \sum_{k=0}^{n-1} \binom{n-1}{k} a^{n-1-k} b^k (1+\mu)(b^*)^k (a^*)^{n-1-k}. \tag{2.45b}$$

Then using (2.40) we conclude that equations (2.44a, 2.44b) hold for all n. Now we are in position to prove that can^{-1} is the inverse of can. Take $x \in P$ and $n \in \mathbb{N}$. Then

$$\begin{aligned}\mathrm{can}(\mathrm{can}^{-1}(x \otimes u^n)) &= \mathrm{can}\left(\sum_{k=0}^{n} \binom{n}{k} x(a^*)^{n-k}(b^*)^k \otimes_B b^k a^{n-k}\right) \\ &= \sum_{k=0}^{n} \binom{n}{k} x(a^*)^{n-k}(b^*)^k b^k a^{n-k} \otimes u^n = x \otimes u^n,\end{aligned}$$

where the last equality follows from (2.44a). Similarly, the use of (2.44b) confirms that

$$\mathrm{can}(\mathrm{can}^{-1}(x \otimes u^{-n})) = x \otimes u^{-n}.$$

Conversely we need to check the equality $\mathrm{can}^{-1}(\mathrm{can}(x \otimes_B y)) = x \otimes_B y$ for all $x, y \in P$. Since P is a $\mathbb{Z}$-graded algebra, it suffices to take homogeneous y of degree n. Suppose $n \geq 0$. Then

$$\mathrm{can}^{-1}((\mathrm{can}(x \otimes_B y)) = \mathrm{can}^{-1}(xy \otimes u^n) = \sum_{k=0}^{n} \binom{n}{k} xy(a^*)^{n-k}(b^*)^k \otimes_B b^k a^{n-k}.$$

Since $\deg(y) = n$, each of the $y(a^*)^{n-k}(b^*)^k$ has degree 0, hence it is in B and we can write

$$\mathrm{can}^{-1}((\mathrm{can}(x \otimes_B y)) = \sum_{k=0}^{n} \binom{n}{k} x \otimes_B y(a^*)^{n-k}(b^*)^k b^k a^{n-k} = x \otimes_B y,$$

by (2.44a). In the case of homogeneous y of negative degree, we use equation (2.44b) to obtain the assertion. Thus we have proven that can is a bijective map, i.e., $A^\infty(S^3_\mu)$ is a Hopf-Galois extension of $A^\infty(S^2_\mu)$ as claimed.

2.4.3 Monopole connection and projectors of charge n

The antipode of the Hopf algebra H of functions on $U(1)$ is involutive, i.e. $S \circ S = H$, hence, in particular, invertible, and so $P(B)^H$ is a symmetric H-Hopf Galois extension. Hence it makes sense to seek the strong connection form $\ell : H \to P \otimes P$ (Definition 2.2.3).

The left coaction (1.33) for $P = A^\infty(S^3_\mu)$ comes out in terms of the $\mathbb{Z}$-grading as

$${}^H\rho(x) = u^{-\deg(x)} \otimes x,$$

for any homogeneous $x \in P$. We define the strong connection form $\ell : H \to P \otimes P$ by

$$\ell(u^n) = \sum_{k=0}^{n} \binom{n}{k} (a^*)^{n-k}(b^*)^k \otimes b^k a^{n-k}, \tag{2.46a}$$

$$\ell(u^{-n}) = (1 + n\mu)^{-1} \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \otimes (b^*)^k (a^*)^{n-k}, \tag{2.46b}$$

$$\ell(1) = 1 \otimes 1, \tag{2.46c}$$

for $n \in \mathbb{N}$. Note that this is simply the expression for $\mathrm{can}^{-1}(1 \otimes h)$ lifted to $P \otimes P$ by omitting the decoration $_B$ on $\otimes_B$ (cf. equation (2.43)), and therefore similar arguments to those used to prove that can^{-1} is the right inverse of can ensure that the equation (2.18) is satisfied. Next, by the definition of ℓ, the equation (2.17) is satisfied ($e = 1$). Moreover, similarly as in the discussion after the definition of can^{-1}, by counting the degree, it follows that ℓ is a right H-colinear map, hence the equation (2.20) holds. Finally, since $\deg((a^*)^{n-k}(b^*)^k) = -n$ and $\deg(a^{n-k}b^k) = n$ one easily realises that the map ℓ is also left H-colinear, so that the equation (2.19) is satisfied. Thus we have constructed a strong connection in the quantum contact Hopf fibration with the strong connection form ℓ.

In the case of the Hopf-Galois extension $A^\infty(S^3_\mu)(A^\infty(S^2_\mu))^H$, each of the u^n is a grouplike element. Therefore the map $\ell = \sum_{i=0}^n l(u^n)_i \otimes r(u^n)_i$ defined by formulae (2.46) gives an infinite family of idempotents $E(u^n)$, $n \in \mathbb{Z}$, defined using the formula (2.32). Each of the $E(u^n)$ is an $(|n|+1) \times (|n|+1)$-matrix with entries from $A^\infty(S^2_\mu)$ (the latter claim can be easily confirmed by the degree counting). Obviously there is an ambiguity in factorising ℓ into $\sum_{i=0}^n l(u^n)_i \otimes r(u^n)_i$ (scalar coefficients can be factorised in infinitely many ways into legs of a tensor product). However if one requires $E(u^n)$ to be Hermitian (i.e. projectors in B) then the unique possibility turns out to be

$$\ell(1) = 1 \otimes 1,$$

$$\ell(u^n) = \sum_{k=0}^{n} \left[\sqrt{\binom{n}{k}}(a^*)^{n-k}(b^*)^k\right] \otimes \left[\sqrt{\binom{n}{k}}b^k a^{n-k}\right], \tag{2.47a}$$

$$\ell(u^{-n}) = \sum_{k=0}^{n} \left[\sqrt{\binom{n}{k}}(1+n\mu)^{-1}a^{n-k}b^k\right] \otimes \left[\sqrt{\binom{n}{k}}(b^*)^k(a^*)^{n-k}\right], \tag{2.47b}$$

for $n \in \mathbb{N}$. This choice leads to an infinite family of Hermitian projectors with entries from $A^\infty(S^2_\mu)$. Explicitly, $E(1) = 1$, and

$$E(u^n)_{kl} = \sqrt{\binom{n}{k}\binom{n}{l}}\, b^k a^{n-k}(a^*)^{n-l}(b^*)^l, \tag{2.48a}$$

$$E(u^{-n})_{kl} = \sqrt{\binom{n}{k}\binom{n}{l}}\, (b^*)^k(a^*)^{n-k}(1+n\mu)^{-1}a^{n-l}b^l, \tag{2.48b}$$

for all $n \in \mathbb{N}$. At this point it is interesting to mention that apparently these formulae are polynomial in a, b, a^*, b^*, and polynomial only in $(1+n\mu)^{-1}$, with $n \in \mathbb{N}$, but not in μ (e.g. (2.48b)). However, their entries have to be properly rearranged in order to express them in terms of the generators X, Z, Z^*. In $A^\infty(S^3_\mu)$ this can be always done with the help of relations (2.37) and (2.40) at the cost of creating new expressions $(1+n\mu)^{-1}$ in (2.48a) and $(1-n\mu)^{-1}$ in (2.48b), with $n \in \mathbb{N}$. Altogether the whole set of projectors can be rewritten in terms of X, Z, Z^* and all $(1+k\mu)^{-1}$ with $k \in \mathbb{Z}$. (The reason for this behaviour is that the elements X, Z, Z^* and μ do not generate the whole of grade-zero polynomial subalgebra of $A(S^3_\mu)$.) For instance the first few projectors $E(u)$, $E(u^{-1})$, $E(u^2)$, $E(u^{-2})$ come out in a matrix form as

$$E(u) = \begin{pmatrix} \frac{1}{2}(1+\mu) + X & Z \\ Z^* & \frac{1}{2}(1+\mu) - X \end{pmatrix}, \tag{2.49a}$$

$$E(u^{-1}) = \begin{pmatrix} \frac{1}{2}(1-\mu) + X & Z^* \\ Z & \frac{1}{2}(1-\mu) - X \end{pmatrix}, \tag{2.49b}$$

$$E(u^2) = \frac{1}{1+\mu} \times \begin{pmatrix} \left(X+\frac{1+\mu}{2}\right)\left(X+\frac{1+3\mu}{2}\right) & \sqrt{2}\left(X+\frac{1+3\mu}{2}\right)Z & Z^2 \\ \sqrt{2}Z^*\left(X+\frac{1+3\mu}{2}\right) & 2\left(\frac{1+\mu}{2}+X\right)\left(\frac{1+\mu}{2}-X\right) & \sqrt{2}\left(\frac{1+\mu}{2}-X\right)Z \\ (Z^*)^2 & \sqrt{2}Z^*\left(\frac{1+\mu}{2}-X\right) & \left(\frac{1+\mu}{2}-X\right)\left(\frac{1+3\mu}{2}-X\right) \end{pmatrix}, \tag{2.49c}$$

$$E(u^{-2}) = \frac{1}{1-\mu} \times \begin{pmatrix} \left(X+\frac{1-\mu}{2}\right)\left(X+\frac{1-3\mu}{2}\right) & \sqrt{2}\left(X+\frac{1-3\mu}{2}\right)Z^* & (Z^*)^2 \\ \sqrt{2}Z\left(X+\frac{1-3\mu}{2}\right) & 2\left(\frac{1-\mu}{2}+X\right)\left(\frac{1-\mu}{2}-X\right) & \sqrt{2}\left(\frac{1-\mu}{2}-X\right)Z^* \\ Z^2 & \sqrt{2}Z\left(\frac{1-\mu}{2}-X\right) & \left(\frac{1-\mu}{2}-X\right)\left(\frac{1-3\mu}{2}-X\right) \end{pmatrix}. \tag{2.49d}$$

Note an interesting symmetry between $E(u^n)$ and $E(u^{-n})$ for low values of n. The projector $E(u^{-n})$ is obtained from the projector $E(u^n)$ by replacing μ by $-\mu$ and interchanging of Z with Z*. This is true for any value of charge n as can be verified directly from the explicit expressions for greater charges n, which can be presented. However this follows also from the following symmetry properties. First observe that the transformation

$$Z \mapsto Z^*, \quad Z^* \mapsto Z, \quad X \mapsto X, \quad \mu \mapsto -\mu, \tag{2.50}$$

does not affect the defining relations (2.42) and defines an automorphism of the algebra $A^\infty(S^2_\mu)$. This symmetry of $A^\infty(S^2_\mu)$ comes in fact from the following symmetry of $A^\infty(S^3_\mu)$. Using the elements $\sqrt{1+k\mu} \in A^\infty(S^3_\mu)$, their inverses $1/\sqrt{1+k\mu}$ for $k \in \mathbb{Z}$ and the relations (2.41) we see that the map

$$\Theta : A^\infty(S^3_\mu) \to A^\infty(S^3_\mu), \quad \mu \mapsto -\mu, \quad a \mapsto A = \sqrt{1-\mu}a^*, \quad b \mapsto B = \sqrt{1-\mu}b^*, \tag{2.51}$$

extends to an algebra automorphism. Note that Θ maps degree n elements to degree $-n$ elements, and on the level of degree 0 elements corresponds to the automorphism of $A^\infty(S^2_\mu)$ given by (2.50). Now, using the relations

$$a^n(1-\mu) = \frac{1+(n-1)\mu}{1+n\mu}a^n, \quad b^n(1-\mu) = \frac{1+(n-1)\mu}{1+n\mu}b^n, \quad n > 0 \tag{2.52}$$

and the fact that μ is central in $A^\infty(S^2_\mu)$ we find that

$$\Theta(p(u^n)_{kl}) = E(u^{-n})_{kl}. \tag{2.53}$$

Thus the authomorphism $\Theta|_{A^\infty(S^2_\mu)} : A^\infty(S^2_\mu) \to A^\infty(S^2_\mu)$ turns the degree n projectors into degree $-n$ projectors, and its existence proves the stated symmetry of monopole projectors.

2.4.4 Remarks about adjoining elements and representations

If μ is not required to be invertibile the underlying polynomial $*$-algebra $A(S^3_\mu)$ admits a specification to $\mu = 0$, after which it coincides with the usual $*$-algebra of polynomials on the 'classical' S^3. As far as the polynomial $*$-algebra $A'(S^2_\mu)$ is concerned, since μ is central, it can be specified to any nonzero real value. This yields a family of quantum 2-spheres isomorphic to the universal enveloping algebra of $su(2)$ with a constrained value $1/\mu^2$ of the quadratic Casimir element (cf. [20]). The $*$-algebra $A(S^2_\mu)$ (i.e., when the invertibility of μ is not assumed) admits in addition a specification to $\mu = 0$, which clearly corresponds to polynomials on the 'classical' S^2. Note that the equations (2.48) with μ considered as a real deformation parameter rather than as a central generator define also a family of projectors $p(u^n)$ over such quantum 2-spheres. In particular, when $\mu = 0$, the projectors $p(u^n)$ correspond to the line bundle projectors of the monopole charge n over S^2. Note also, that then Θ defined by (2.51) is the orientation reversing automorphism of S^3.

It can be seen that the $*$-algebras $A(S^3_\mu)$ and $A(S^2_\mu)$ admit certain C^*-algebraic completions. Although intuitively resembling some topological quantum four (resp. three) dimensional spaces rather than 3-spheres (resp. 2-spheres), they nevertheless would constitute interesting examples. Unfortunately, there is one obstacle for this task. In order to write our formulae (2.43) for the inverse of the canonical map and then for the connection and the projectors we should adjoin to $A(S^3_\mu)$ and to $A(S^2_\mu)$ an infinite number of elements $(1+k\mu)^{-1}$, $k \in \mathbb{Z}$. (To be able to implement the symmetry discussed at the end of Section 2.4.3, we should adjoin additionally the elements $\sqrt{1+k\mu}$ and $1/\sqrt{1+k\mu}$.) It can be seen that this spoils not only the C^*-algebraic completion but even the $*$-algebraic version. This follows from the representation theory of e.g. $A(S^2_\mu)$, which can be inferred from that of $su(2)$. In fact all bounded representations decompose into finite dimensional irreducible ones. These in turn are as follows. There is a family of one-dimensional representations (characters) parametrised by the points of S^2, which represent μ by 0 (they obviously do not extend to $A'(S^2_\mu)$). In addition, in each dimension $N \in \mathbb{N}$ there are two $*$-representations, labelled by $\sigma = \pm 1$, which represent μ either by $1/N$ or by $-1/N$. Namely, they are given by

$$\begin{aligned}
\mu\, v_m &= \tfrac{\sigma}{N}\, v_m, \\
X\, v_m &= \tfrac{\sigma m}{N}\, v_m, \\
Z\, v_m &= \tfrac{\sigma}{2N}\sqrt{(N+1-2m)(N-1+2m)}\, v_{m-1}, \\
Z^*\, v_m &= \tfrac{\sigma}{2N}\sqrt{(N-1-2m)(N+1+2m)}\, v_{m+1},
\end{aligned} \tag{2.54}$$

with respect to an orthonormal basis v_m, where $m \in \{-\frac{N-1}{2}, -\frac{N-3}{2}, \ldots, \frac{N-3}{2}, \frac{N-1}{2}\}$.

This excludes a possibility of adjoining the needed elements if we want our C^*-algebra to describe more than merely the commutative S^2. At this point it is interesting to make the following observation. Had we constrained ourselves to construct just certain subclass of projectors, say for a selected class of charges, we might have a chance to accomplish a nondegenerate $*$-algebra extended by some (but not all) elements $(1+k\mu)^{-1}$ and then also its C^*-algebraic version. For instance restricting only to the positive charges n leaves at our disposal the series of all representations that represent μ by $1/N$. It is not clear however what could be a possible interpretation of such a breaking of the (magnetic) charge conjugation by the contact structure quantisation with a noncentral parmeter.

2.5 Cleft extensions

An interesting class of coalgebra-Galois extensions is provided by cleft extensions. These can be characterised as such extensions that every associated module (in the sense of Definition 2.3.1) is a free module.

Definition 2.5.1. Let C be a coalgebra, P be an algebra and a right C-comodule. A convolution invertible and a right C-colinear map $\gamma : C \to P$ is called a *cleaving map*. A C-coalgebra Galois extension $P(B)^C$ such that there exists a cleaving map $\gamma : C \to P$ is called a *cleft coalgebra Galois extension* and is denoted by $P(B)^C_\gamma$. Similarly a *cleft* $(P,C)_\psi$*-extension* $P_\gamma(B,C,\psi)$ is a $(P,C)_\psi$-extension with a cleaving map γ.

Observe that if $P(B)^C$ is cleft, then the inverse of the canonical map has the form

$$(\mathrm{can}_P^C)^{-1}(p\otimes c) = p\gamma^{-1}(c_{(1)})\otimes_B \gamma(c_{(2)}),\ \text{for all } c\in C, p\in P, \tag{2.55}$$

where γ^{-1} means the convolution inverse.

Lemma 2.5.2. *(cf. [3], proof of the Proposition 2.3) If $P_\gamma(B,C,\psi)$ is a cleft $(P,C)_\psi$-extension then, for any $c\in C$,*

$$\psi(c_{(1)}\otimes\gamma^{-1}(c_{(2)})) = \gamma^{-1}(c)\rho^C(1_P), \tag{2.56}$$

i.e., if ψ is bijective, then γ^{-1} is left C-colinear with respect to the left C-coaction (1.32). Moreover if C is a Hopf algebra and P is a right C-comodule algebra, then, for any cleaving map $\gamma : C\to P$ and $c\in C$,

$$\rho^C(\gamma^{-1}(c)) = \gamma^{-1}(c_{(2)})\otimes Sc_{(1)}. \tag{2.57}$$

Observe that if $P(B)^C_\gamma$ is a cleft C-Hopf Galois extension and the antipode in C is bijective, then statements (2.56) and (2.57) are mutually equivalent.

Note that if $P(B)^C_{\gamma,e}$ is a symmetric e-copointed cleft C-coalgebra Galois extension, then, by (2.57), it is a principal extension, with a strong connection form

$$\ell : C\to P\otimes P,\ \ c\mapsto \gamma^{-1}(c_{(1)})\otimes\gamma(c_{(2)}). \tag{2.58}$$

Proposition 2.5.3. *(Proposition 2.3 [3].) Let C be a coalgebra, P be a right comodule and $B = P^{coC}$. If there exists a cleaving map $\gamma : C \to P$, then the following are equivalent:*

1. *$P(B)^C$ is a C-coalgebra Galois extension.*
2. *There exists an entwining ψ such that $P(B, C, \psi)$ is a $(P, C)_\psi$-extension.*
3. *For all $p \in P$, $p_{(0)}\gamma^{-1}(p_{(1)}) \in B$.*

If any of the above conditions hold, then $P \simeq B \otimes C$ in ${}_B\mathcal{M}^C$, where the isomorphism $\theta_\gamma : P \to B \otimes C$ and its inverse $\theta_\gamma^{-1} : B \otimes C \to P$ are given explicitly by

$$\begin{aligned} \theta_\gamma(p) &= p_{(0)}\gamma^{-1}(p_{(1)}) \otimes p_{(2)}, \\ \theta_\gamma^{-1}(b \otimes c) &= b\gamma(c). \end{aligned} \tag{2.59}$$

Lemma 2.5.4. *Assume that P is an (H, C)-bicomodule and H-comodule algebra, where C is a coalgebra and H is a Hopf algebra. If there exists a coalgebra map $f : C \to H$ and a cleaving map $\gamma : C \to P$, such that, for any $c \in C$,*

$${}^H\rho(\gamma(c)) = f(c_{(1)}) \otimes \gamma(c_{(2)}), \tag{2.60}$$

then, for any $c \in C$,

$${}^H\rho(\gamma^{-1}(c)) = Sf(c_{(2)}) \otimes \gamma^{-1}(c_{(1)}), \tag{2.61}$$

$$\gamma^{-1}(c_{(1)})_{(-1)}\gamma(c_{(2)})_{(-1)} \otimes \gamma^{-1}(c_{(1)})_{(0)} \otimes \gamma(c_{(2)})_{(0)} = 1_H \otimes \gamma^{-1}(c_{(1)}) \otimes \gamma(c_{(2)}). \tag{2.62}$$

Proof. Applying

$$(m \otimes P) \circ (H \otimes T_{P,H}) \circ (H \otimes m \otimes H) \circ ({}^H\rho \otimes \gamma^{-1} \otimes (S \circ f)),$$

where $T_{P,H} : P \otimes H \ni p \otimes h \mapsto h \otimes p \in H \otimes P$ is a flip map, to the both sides of the identity

$$\gamma^{-1}(c_{(1)})\gamma(c_{(2)}) \otimes c_{(3)} \otimes c_{(4)} = 1_P \otimes c_{(1)} \otimes c_{(2)},$$

yields (2.61). Then

$$\begin{aligned} &\gamma^{-1}(c_{(1)})_{(-1)}\gamma(c_{(2)})_{(-1)} \otimes \gamma^{-1}(c_{(1)})_{(0)} \otimes \gamma(c_{(2)})_{(0)} \\ &\qquad = Sf(c_{(2)})f(c_{(3)}) \otimes \gamma^{-1}(c_{(1)}) \otimes \gamma(c_{(4)}) = 1_H \otimes \gamma^{-1}(c_{(1)}) \otimes \gamma(c_{(2)}). \end{aligned}$$

□

Lemma 2.5.5. *Suppose that $P(B)^C_\gamma$ is a cleft C-coalgebra Galois extension, and that $\gamma : C \to P$ is a cleaving map on P. The map*

$$\gamma' = m \circ (\Gamma \otimes \gamma) \circ \Delta, \tag{2.63}$$

where $\Gamma : C \to B$ is a convolution invertible map called a gauge transformation, *is also a cleaving map on P, and any other cleaving map on P has this form.*

2.5.1 The quantum solid torus

The quantum solid torus is an example of a cleft Hopf-Galois extension.

Denote by h the unitary and central generator of the coordinate algebra of the unit circle $\vartheta(S^1)$.

Solid torus is the Cartesian product $D \times S^1$ of the unit disc and the unit circle. Therefore, one can define a coordinate algebra of quantum torus as the tensor product $\vartheta(D_p) \otimes \vartheta(S^1)$ of the coordinate algebra of the quantum unit disc eq. (1.4) and the coordinate algebra of the unit circle. One can introduce a further quantisation parameter by making the tensor product noncommutative, i.e., by requesting that the subalgebras $\vartheta(D_p) \otimes 1$ and $1 \otimes \vartheta(S^1)$ do not mutually commute. We identify the generators $x \otimes 1$ and $1 \otimes h$ with x and h respectively. The coordinate algebra of the quantum solid torus $\vartheta(D_p \times_\theta S^1)$, $0 < p < 1$, $0 \leq \theta < 2\pi$, is generated as a $*$-algebra by x and h, subject to the relations

$$\begin{aligned} hh^* = 1 = h^*h, \;\; x^*x - pxx^* = 1 - p, \\ hx = e^{i\theta}xh, \;\; hx^* = e^{-i\theta}x^*h. \end{aligned} \tag{2.64}$$

The linear basis of $\vartheta(D_p \times_\theta S^1)$ consists of the elements of the form

$$(1 - xx^*)^k x^m h^n, \quad k \in \mathbb{N}_0, \; m, n \in \mathbb{Z}. \tag{2.65}$$

Observe that there exists a surjective $*$-algebra morphism

$$\pi_\partial : \vartheta(D_p \times_\theta S^1) \to \vartheta(T_\theta), \; \pi_\partial(h) = U, \; \pi_\partial(x) = V, \tag{2.66}$$

of the quantum solid torus onto the quantum torus (eq. (1.15)), i.e., the quantum torus is the boundary of the quantum solid torus.

Enveloping C^*-algebra $C(D_p \times_\theta S^1)$ can be obtained from $\vartheta(D_p \times_\theta S^1)$ using C^* representations. It is possible as $\|x\| = 1$ and $\|h\| = 1$ for any representation, and hence the norms of polynomials in h and x are bounded for all representations. Irreducible representations of $\vartheta(D_p \times_\theta S^1)$ include those unitarily isomorphic either to the representation obtained by composing one of the irreducible representations of the quantum torus $\vartheta(T_\theta)$ with the map π_∂ as well one of the representations $\varrho^\alpha_{p,\theta} : \vartheta(D_p \times_\theta S^1) \to \mathbf{B}(\mathcal{H}^\alpha_{p,\theta})$, where $\mathcal{H}^\alpha_{p,\theta}$ is generated by orthonormal vectors Ψ_n, $n \in \mathbb{N}_0$, and

$$\begin{aligned} \varrho^\alpha_{p,\theta}(x)\Psi_n = \sqrt{1 - p^{n+1}}\Psi_{n+1}, \;\; \varrho^\alpha_{p,\theta}(x^*)\Psi_n = \sqrt{1 - p^n}\Psi_{n-1} \text{ if } n > 0, \\ \varrho^\alpha_{p,\theta}(x^*)\Psi_0 = 0, \;\; \varrho^\alpha_{p,\theta}(h^{\pm 1}) = e^{\pm i(\alpha + n\theta)}\Psi_n. \end{aligned} \tag{2.67}$$

Denote by $H = \vartheta(U(1))$ the coordinate algebra of $U(1)$ and let u be the unitary generator of H. The algebra $\vartheta(D_p \times_\theta S^1)$ is clearly a right H-comodule $*$-algebra, with the coaction defined on the generators by

$$\rho^H(x) = x \otimes 1, \;\; \rho^H(h) = h \otimes u. \tag{2.68}$$

It is easy to see that, $\vartheta(D_p \times_\theta S^1)^{\mathrm{co}H} = \vartheta(D_p)$ and $\vartheta(D_p \times_\theta S^1)(\vartheta(D_p))^H_{\gamma_T}$ is a cleft H-Hopf Galois extension, where, for all $n \in \mathbb{Z}$,

$$\gamma_T : H \to \vartheta(D_p \times_\theta S^1), \;\; u^n \mapsto h^n, \tag{2.69}$$
$$\gamma_T^{-1} : H \to \vartheta(D_p \times_\theta S^1), \;\; u^n \mapsto h^{-n}, \tag{2.70}$$

are the cleaving map and its convolution inverse, respectively.

Chapter 3

Cotensor products of quantum principal bundles.

3.1 Introduction

This chapter presents a method of constructing new quantum principal bundles from the given ones, using a cotensor product. The method was inspired by a recent proposal for the development of homotopy theory of general Hopf-Galois extensions in [26].

Let $M \in \mathcal{M}^C$ and $N \in {}^C\mathcal{M}$. The *cotensor product* $M\Box_C N$ is defined by the following exact sequence:

$$0 \longrightarrow M\Box_C N \longrightarrow M \otimes N \underset{M\otimes {}^C\rho}{\overset{\rho^C \otimes N}{\rightrightarrows}} M \otimes C \otimes N. \tag{3.1}$$

The general idea of the method is as follows. Let C be a coalgebra and H a Hopf algebra. Assume that A is an H-Hopf Galois extension of B, and P is a C-coalgebra Galois extension of D and also an (H,C)-bicomodule and an H-comodule algebra. One can form the cotensor product $A\Box_H P$ of A and P. It is easy to see that $A\Box_H P$ is an algebra and a right C-comodule. It turns out that, under a number of not very restrictive conditions, $A\Box_H P$ is a C-coalgebra Galois extension of $A\Box_H D$.

Classically, the cotensor product of two algebras of functions on compact Hausdorff topological spaces corresponds to the algebra of functions on the Cartesian product of spaces modulo a group action. Note that the discussion here is not meant to be mathematically rigorous. Explicitly, let $A = \vartheta(X)$, $P = \vartheta(Y)$, $H = \vartheta(G)$, where G is a topological group, which acts on the right on a topological space X, and on the left on a topological space Y, so that H is a Hopf algebra, A is a right H-comodule algebra and P is a left H-comodule algebra. Then the group G acts on the right on the Cartesian product $X \times Y$:

$$(X \times Y) \times G \to X \times Y, \ \ ((x,y),g) \mapsto (xg, g^{-1}y),$$

and $A\Box_H P = \vartheta((X \times Y)/G)$.

Suppose, that the action of G on X is free (i.e., that X is a principal bundle). Let $M = X/G$ be the base space of this bundle, and let $\pi : X \to M$ be a natural projection. Then X is a locally trivial fibre bundle, i.e., there exists an open cover $(U_\alpha)_{\alpha\in I}$ of M, such that $\pi^{-1}(U_\alpha)$ is homeomorphic to $U_\alpha \times G$.

On the other hand, it is easy to see that, the surjection

$$\tilde{\pi} : (X \times Y)/G \to M, \quad (x,y) \mapsto \pi(x),$$

is well defined, and that, $\tilde{\pi}^{-1}(U_\alpha) = U_\alpha \times Y$. Hence $(X \times Y)/G$ is a locally trivial fibre bundle, with a base space M and a fibre Y.

Suppose that another group K acts on Y on the right, and that the actions of G and K on Y commute (i.e., dually, for $C = \vartheta(K)$, P is an (H,C)-bicomodule). Assume also that the action of K on Y is free, i.e., Y is also a principal bundle with a structure group K. Under certain conditions, the natural action of K on $(X \times Y)/G$:

$$((x,y),k) \mapsto (x,yk),$$

is also free, hence $(X \times Y)/G$ is a principal bundle with the structure group K and the base space $(X \times (Y/K))/G$. Then we can view $(X \times Y)/G$ in two ways as a tower of fibre bundles. Firstly, it is a fibre bundle with the base space M and the fibre Y, which, in turn, is also a fibre bundle with the base Y/K and the fibre K. Secondly, it is a fibre bundle with the fibre K and the base $(X \times (Y/K))/G$, which in turn is also a fibre bundle with the base M and the fibre Y/K.

The construction of the cotensor product of quantum principal bundles can be thought of as a natural generalisation of prolongations of Hopf-Galois extensions (cf. [38]).

Let A be an H-Hopf Galois extension of B and let P be a Hopf algebra. Suppose that $f : P \to H$ is a surjective Hopf algebra morphism. Define a left H-coaction on P by $p \mapsto f(p_{(1)}) \otimes p_{(2)}$. Then (with certain assumptions about flatness over $\mathbb{K}$) $A\Box_H P$ is a P-Hopf Galois extension of B called a P-prolongation of A. As P is a P-Hopf Galois extension of the ground ring, this is a special case of the cotensor product of quantum principal bundles.

Another special case of the construction described in this chapter is the cotensor product of bigalois objects (cf. [37]).

3.2 The inverse of the canonical map for the cotensor product of quantum principal bundles.

Lemma 3.2.1. *If C is a coalgebra, flat as a $\mathbb{K}$-module, P is an (H,C)-bicomodule and A is a right H-comodule, then $A\Box_H P$ is a right C-comodule. Moreover, if H is a bialgebra and A and P are H-comodule algebras, then $A\Box_H P$ is an algebra.*

Proof. The first statement is the standard result in the coalgebra theory (cf. 11.3 [13]). We include its proof for completeness.

There exists a natural right C-comodule structure on $A \otimes P$, given by the coaction on the second factor, $\rho^C_{A\otimes P} : a \otimes p \mapsto a \otimes \rho^C_P(p)$. Let

$$\tilde{\rho}^C_{A\Box_H P} = \rho^C_{A\otimes P}\Big|_{A\Box_H P} : A\Box_H P \to A \otimes P \otimes C.$$

By the flatness of C over $\mathbb{K}$, the exactness of the defining sequence,

$$0 \longrightarrow A\Box_H P \longrightarrow A \otimes P \underset{A\otimes {}^H\rho_P}{\overset{\rho^H_A \otimes P}{\rightrightarrows}} A \otimes H \otimes P, \tag{3.2}$$

implies the exactness of the top row of the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & (A\Box_H P)\otimes C \arrow[r] & A \otimes P \otimes C \arrow[r, shift left, "\rho^H_A\otimes P\otimes C"] \arrow[r, shift right, "A\otimes {}^H\rho_P\otimes C"'] & A\otimes H\otimes P\otimes C . \\ & & A\Box_H P \arrow[u, "\tilde{\rho}^C_{A\Box_H P}"'] \arrow[ul, dashed, "\rho^C_{A\Box_H P}"] & \end{tikzcd} \tag{3.3}$$

As P is an (H, C)-bicomodule,

$$(\rho^H_A \otimes P \otimes C - A \otimes {}^H\rho_P \otimes C) \circ \tilde{\rho}^C_{A\Box_H P}$$
$$= (A \otimes H \otimes \rho^C_P) \circ (\rho^H_A \otimes P - A \otimes {}^H\rho_P)\Big|_{A\Box_H P} = 0,$$

by the definition of $A\Box_H P$. Hence, by the universal property of the kernel, there is a unique factorisation $\rho^C_{A\Box_H P}$ (cf. diag. (3.3)).

One defines the multiplication on $A\Box_H P$ first by restricting the usual tensor product multiplication $(a \otimes p) \otimes (a' \otimes p') \mapsto (aa' \otimes pp')$ to $\tilde{m}_{A\Box_H P} : (A\Box_H P) \otimes (A\Box_H P) \to (A \otimes P)$. Then, using that A and P are H-comodule algebras, one proves that it gives zero when composed with the equalising map, so there exists a unique factorisation $m_{A\Box_H P} : (A\Box_H P) \otimes (A\Box_H P) \to (A\Box_H P)$. □

The following lemma is probably well-known in ring and module theory. However, since we were not able to find an exact reference, we carefully provide an explicit proof.

Lemma 3.2.2. *Suppose that A and B are algebras, M is a right A-module and N is a right B-module. Then $M \otimes N$ is a right $A \otimes B$ module, with the obvious tensor action $(m \otimes n) \cdot (a \otimes b) \mapsto ma \otimes nb$. On the other hand, if P is a left $A \otimes B$-module, it is also a left A- and B-module, with the left A-action $a \cdot p \mapsto (a \otimes 1_B)p$, and the left B-action $b \cdot p \mapsto (1_P \otimes b)p$. Obviously, these A- and B-actions commute. Furthermore $(N \otimes_B P) \in {}_A\mathcal{M}$, with the left A-action $a \cdot (n \otimes_B p) \mapsto n \otimes_B (ap)$. The following statements are true:*

1. *For all P in ${}_{A\otimes B}\mathcal{M}$, the map*

$$\phi_P : M \otimes_A (N \otimes_B P) \to (M \otimes N) \otimes_{A\otimes B} P,$$
$$m \otimes_A (n \otimes_B p) \mapsto (m \otimes n) \otimes_{A\otimes B} p \tag{3.4}$$

is a $\mathbb{K}$-linear isomorphism, with the inverse given explicitly by:

$$\begin{aligned} \phi_P^{-1} : (M \otimes N) \otimes_{A\otimes B} P &\to M \otimes_A (N \otimes_B P), \\ (m \otimes n) \otimes_{A\otimes B} p &\mapsto m \otimes_A (n \otimes_B p). \end{aligned} \tag{3.5}$$

2. *If M is a flat right A-module and N is a flat right B-module, then $M \otimes N$ is a flat right $A \otimes B$-module.*

3. *If M is a faithfully flat right A-module and N is a faithfully flat right B-module, then $M \otimes N$ is a faithfully flat right $A \otimes B$-module.*

Proof. **1.** First we show that ϕ_P is well defined. Indeed, consider the map

$$\begin{aligned} \overline{\overline{\phi_P}} : M \otimes (N \otimes P) &\to (M \otimes N) \otimes_{A\otimes B} P, \\ m \otimes (n \otimes p) &\mapsto (m \otimes n) \otimes_{A\otimes B} p. \end{aligned}$$

The map $\overline{\overline{\phi_P}}$ is the composition of the associativity isomorphism from the tensor product over $\mathbb{K}$ with the canonical surjection from the tensor product over $\mathbb{K}$ to the tensor product over $A \otimes B$, hence it is well defined.

Since the tensor product functor is right exact, the top row in the following diagram is exact, as it is obtained by applying the functor $M \otimes \cdot$ to the exact sequence defining the tensor product over B.

$$\begin{tikzcd} M \otimes (N \otimes B \otimes P) \arrow[r, shift left, "M\otimes\rho_B\otimes P"] \arrow[r, shift right, "M\otimes N\otimes_B\rho"'] & M \otimes (N \otimes P) \arrow[r] \arrow[d, "\overline{\overline{\phi_P}}"'] & M \otimes (N \otimes_B P) \arrow[r] \arrow[dl, dashed, "\overline{\phi_P}"] & 0. \\ & (M \otimes N) \otimes_{A\otimes B} P & & \end{tikzcd} \tag{3.6}$$

By the universal property of a cokernel, in order to prove that $\overline{\overline{\phi_P}}$ factorises uniquely through $\overline{\phi_P}$ and the canonical surjection onto the tensor product over B, it is enough to show that

$$\overline{\overline{\phi_P}} \circ (M \otimes \rho_B \otimes P) = \overline{\overline{\phi_P}} \circ (M \otimes N \otimes {}_B\rho).$$

Indeed, for all $m \otimes (n \otimes b \otimes p) \in M \otimes (N \otimes B \otimes P)$,

$$\begin{aligned} \overline{\overline{\phi_P}} \circ (M \otimes \rho_B \otimes P)(m \otimes (n \otimes b \otimes p)) &= \overline{\overline{\phi_P}}(m \otimes (nb \otimes p)) \\ &= (m \otimes nb) \otimes_{A\otimes B} p = (m \otimes n)(1_A \otimes b) \otimes_{A\otimes B} p \\ &= (m \otimes n) \otimes_{A\otimes B} (1_A \otimes b)p = (m \otimes n) \otimes_{A\otimes B} bp \\ &= \overline{\overline{\phi_P}}(m \otimes (n \otimes bp)) = \overline{\overline{\phi_P}} \circ (M \otimes N \otimes {}_B\rho)(m \otimes (n \otimes b \otimes p)). \end{aligned}$$

Similarly, (in the diagram below, with $Q = N \otimes_B P$),

$$\begin{tikzcd} M \otimes A \otimes Q \arrow[r, shift left, "\rho_A\otimes Q"] \arrow[r, shift right, "M\otimes_A\rho"'] & M \otimes Q \arrow[r] \arrow[d, "\overline{\phi_P}"'] & M \otimes_A Q \arrow[r] \arrow[dl, dashed, "\phi_P"] & 0 \\ & (M \otimes N) \otimes_{A\otimes B} P, & & \end{tikzcd}$$

one shows that, for all $m \otimes a \otimes (n \otimes_B p) \in M \otimes A \otimes (N \otimes_B P)$,

$$\begin{aligned}
\overline{\phi_P} \circ (\rho_A \otimes (N \otimes_B P))(m \otimes a \otimes (n \otimes_B p)) &= \overline{\phi_P}(ma \otimes (n \otimes_B p)) \\
= (ma \otimes n) \otimes_{A \otimes B} p &= (m \otimes n)(a \otimes 1_B) \otimes_{A \otimes B} p = (m \otimes n) \otimes_{A \otimes B} ap \\
&= \overline{\phi_P}(m \otimes (n \otimes ap)) = \overline{\phi_P} \circ (M \otimes {}_A\rho)(m \otimes a \otimes (n \otimes_B p)),
\end{aligned}$$

hence the map ϕ_P exists, and it is the unique factorisation of $\overline{\phi_P}$, through the canonical surjection onto the tensor product over B.

In the same way one proves that ϕ_P^{-1} is well defined. Indeed, the map

$$\begin{aligned}
\overline{\phi_P^{-1}} : (M \otimes N) \otimes P &\to M \otimes_A (N \otimes_B P), \\
(m \otimes n) \otimes p &\mapsto m \otimes_A (n \otimes_B p)
\end{aligned} \tag{3.7}$$

is a composition of the associativity bijection for the tensor product over $\mathbb{K}$ with the two surjections onto the tensor product over A and B. Furthermore, for all $(m \otimes n) \otimes (a \otimes b) \otimes p \in (M \otimes N) \otimes (A \otimes B) \otimes P$,

$$\begin{aligned}
&\overline{\phi_P^{-1}} \circ (\rho_{A \otimes B} \otimes P)((m \otimes n) \otimes (a \otimes b) \otimes p) = \overline{\phi_P^{-1}}((ma \otimes nb) \otimes p) \\
&\quad = ma \otimes_A (nb \otimes_B p) = m \otimes_A (n \otimes_B abp) = m \otimes_A (n \otimes_B ((a \otimes b)p)) \\
&= \overline{\phi_P^{-1}}((m \otimes n) \otimes ((a \otimes b)p)) = \overline{\phi_P^{-1}} \circ ((M \otimes N) \otimes {}_{A \otimes B}\rho)((m \otimes n) \otimes (a \otimes b) \otimes p),
\end{aligned}$$

hence the map ϕ_P^{-1} exists and it is a unique factorisation of $\overline{\phi_P^{-1}}$ through the surjection onto the tensor product over $A \otimes B$.

Since, for all $m \otimes_A (n \otimes_B p) \in M \otimes_A (N \otimes_B P)$, $(m' \otimes n') \otimes_{A \otimes B} p' \in (M \otimes N) \otimes_{A \otimes B} P$,

$$\begin{aligned}
\phi_P^{-1} \circ \phi_P(m \otimes_A (n \otimes_B p)) &= \phi_P^{-1}((m \otimes n) \otimes_{A \otimes B} p) = m \otimes_A (n \otimes_B p), \\
\phi_P \circ \phi_P^{-1}((m' \otimes n') \otimes_{A \otimes B} p') &= \phi_P(m' \otimes_A (n' \otimes_B p')) = (m' \otimes n') \otimes_{A \otimes B} p',
\end{aligned}$$

the map ϕ_P is a linear bijection and ϕ_P^{-1} is its inverse.

Let $P, P', P'' \in {}_{A \otimes B}\mathcal{M}$. The following diagram will be used in the remaining part of proof:

$$\begin{tikzcd}
0 \arrow[r] & M \otimes_A (N \otimes_B P) \arrow[r] \arrow[d, "\phi_P"] & M \otimes_A (N \otimes_B P') \arrow[r] \arrow[d, "\phi_{P'}"] & M \otimes_A (N \otimes_B P'') \arrow[r] \arrow[d, "\phi_{P''}"] & 0 \\
0 \arrow[r] & (M \otimes N) \otimes_{A \otimes B} P \arrow[r] & (M \otimes N) \otimes_{A \otimes B} P' \arrow[r] & (M \otimes N) \otimes_{A \otimes B} P'' \arrow[r] & 0.
\end{tikzcd} \tag{3.8}$$

2. Suppose we are given an exact sequence of left $A \otimes B$-modules:

$$0 \longrightarrow P \xrightarrow{f} P' \xrightarrow{g} P'' \longrightarrow 0 \tag{3.9}$$

If the maps in the top row of the diagram (3.8) are $M \otimes_A (N \otimes_B f)$ and $M \otimes_A (N \otimes_B g)$, then, since M and N are flat modules, the top row sequence in (3.8) is exact. If the diagram (3.8) is commutative, then, as the vertical maps are linear bijections, the bottom row is also exact. And the bottom horizontal maps are

$$\begin{aligned} \phi_{P'} \circ (M \otimes_A (N \otimes_B f)) \circ \phi_P^{-1} &= (M \otimes N) \otimes_{A\otimes B} f, \\ \phi_{P''} \circ (M \otimes_A (N \otimes_B g)) \circ \phi_{P'}^{-1} &= (M \otimes N) \otimes_{A\otimes B} g. \end{aligned} \tag{3.10}$$

Hence $M \otimes N$ is a flat $A \otimes B$-module.
3. By the previous part of the lemma, $M \otimes N$ is a flat $A \otimes B$ module. Suppose that, given a, not necessarily exact, sequence of maps (3.9), the induced sequence in the bottom row of diagram (3.8) is exact. Let the maps in the top row be

$$\begin{aligned} \phi_{P'}^{-1} \circ ((M \otimes N) \otimes_{A\otimes B} f) \circ \phi_P &= (M \otimes_A (N \otimes_B f)), \\ \phi_{P''}^{-1} \circ ((M \otimes N) \otimes_{A\otimes B} g) \circ \phi_{P'} &= (M \otimes_A (N \otimes_B g)), \end{aligned}$$

so that the diagram (3.8) is commutative. Since the vertical maps are bijective and the bottom row is exact, the top row is exact as well. Then, using the faithful flatness of M and N, one can reconstruct the sequence (3.9) from the top row of (3.8), knowing it has to be exact. Hence $M \otimes N$ is faithfully flat as a right $A \otimes B$-module. □

The following lemma is a generalisation of a known property of Hopf Galois extensions.

Lemma 3.2.3. *(cf. [38]) Let $A(B)^H$ be an H-Hopf Galois extension (in particular $A \in {}_B\mathcal{M}_B^H$), and let $V \in {}^H\mathcal{M}_D$. If A is faithfully flat as a right B-module and V is a flat right D-module, then $A\square_H V$ is a flat right $B \otimes D$-module. Moreover, if V is a faithfully flat right D-module, then $A\square_H V$ is a faithfully flat right $B \otimes D$-module.*

Proof. $A\square_H V$ is a right $B \otimes D$ module by restriction of the tensor product action $(A \otimes V) \otimes (B \otimes D) \ni (a \otimes v) \otimes (b \otimes d) \mapsto ab \otimes vd \in A \otimes V$. There is a chain of isomorphisms

$$\begin{aligned} A \otimes_B (A\square_H V) \simeq (A \otimes_B A)\square_H V &\xrightarrow{\mathrm{can}_A^H \otimes V} (A \otimes H)\square_H V \simeq A \otimes V, \\ a \otimes_B \sum_i a'_i \otimes v_i &\longmapsto \sum_i aa'_i \otimes v_i. \end{aligned} \tag{3.11}$$

Call this composition f. The leftmost map in the above composition is the natural map

$$A \otimes_B (A\square_H V) \to (A \otimes_B A)\square_H V, \ a \otimes_B (\sum_i a'_i \otimes v_i) \mapsto \sum_i (a \otimes_B a'_i) \otimes v_i,$$

which, by ([41],§1) and the assumption that A is flat as a right B-module, is an isomorphism. Observe that the middle transformation is well defined as can_A^H is a right H-colinear map. View $A \otimes_B (A\square_H V)$ as a right $B \otimes D$-module by the action on the second factor. Then f is a right $B \otimes D$-linear map. The assertions of the lemma easily follow, using the isomorphism f and Lemma 3.2.2. □

Lemma 3.2.4. *Let C be a coalgebra and H be a bialgebra. Let P be an (H,C)-bicomodule and an H-comodule algebra, such that the H-coaction is an algebra map.*

1. *If $P(B)^C$ is a C-coalgebra Galois extension such that $B \subseteq {}^{coH}P$, then, for any $c \in C$,*

$$c^{[1]}{}_{(-1)}c^{[2]}{}_{(-1)} \otimes c^{[1]}{}_{(0)} \otimes_B c^{[2]}{}_{(0)} = 1_H \otimes c^{[1]} \otimes_B c^{[2]}, \tag{3.12}$$

where $c^{[1]} \otimes_B c^{[2]} = (\mathrm{can}_P^C)^{-1}(1 \otimes c)$.

2. *If, in addition, H is a Hopf algebra, then, for any $c \in C$,*

$$c^{[1]}{}_{(-1)} \otimes c^{[1]}{}_{(0)} \otimes_B c^{[2]} = Sc^{[2]}{}_{(-1)} \otimes c^{[1]} \otimes_B c^{[2]}{}_{(0)}. \tag{3.13}$$

Proof. Note that conditions (3.12) and (3.13) make sense. Indeed, view $H \otimes P$ as a (B,B)-bimodule with $b \cdot (h \otimes p) \cdot b' = h \otimes bpb'$. The assumptions that a left H-coaction is an algebra map and $B \subseteq {}^{coH}P$ together imply that the left H-coaction on P is a (B,B)-bimodule map.
1. This follows by applying $(H \otimes (\mathrm{can}_P^C)^{-1}) \circ ({}^H\rho \otimes C)$ to both sides of identity (2.5). Explicitly, take any $c \in C$ and compute

$$\begin{aligned}
&(H \otimes (\mathrm{can}_P^C)^{-1}) \circ ({}^H\rho \otimes C)(c^{[1]}c^{[2]}{}_{(0)} \otimes c^{[2]}{}_{(1)}) \\
&\quad = (H \otimes (\mathrm{can}_P^C)^{-1})(c^{[1]}{}_{(-1)}c^{[2]}{}_{(0)(-1)} \otimes c^{[1]}{}_{(0)}c^{[2]}{}_{(0)(0)} \otimes c^{[2]}{}_{(1)}) \\
&\quad = (H \otimes (\mathrm{can}_P^C)^{-1})(c^{[1]}{}_{(-1)}c^{[2]}{}_{(-1)} \otimes c^{[1]}{}_{(0)}c^{[2]}{}_{(0)(0)} \otimes c^{[2]}{}_{(0)(1)}) \\
&\quad = c^{[1]}{}_{(-1)}c^{[2]}{}_{(-1)} \otimes c^{[1]}{}_{(0)} \otimes_B c^{[2]}{}_{(0)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(H \otimes (\mathrm{can}_P^C)^{-1}) \circ ({}^H\rho \otimes C)(1_P \otimes c) &= (H \otimes (\mathrm{can}_P^C)^{-1})(1_H \otimes 1_P \otimes c) \\
&= 1_H \otimes c^{[1]} \otimes_B c^{[2]},
\end{aligned}$$

hence the required assertion (3.12) follows.
2. Apply $(m \otimes P \otimes P) \circ (S \otimes {}^H\rho \otimes P)$ to both sides of (3.12). □

The following theorem is the main result of this section.

Theorem 3.2.5. *Suppose that:*

1. *$A(B)^H$ is an H-Hopf Galois extension such that A is faithfully flat as a right B-module and flat as a left B-module,*
2. *C is a coalgebra that is flat as a $\mathbb{K}$-module,*
3. *$P(D)^C$ is a C-coalgebra Galois extension, such that P is a faithfully flat right D-module and a flat left D-module,*
4. *P is an (H,C)-bicomodule and an H-comodule algebra,*
5. *$D \subseteq {}^{coH}P$.*

Then $(A\Box_H P)(B\otimes D)^C$ is a C-coalgebra Galois extension, and $A\Box_H P$ is a faithfully flat right $B\otimes D$-module. Explicitly, the inverse of the canonical map is given by:

$$
\begin{aligned}
&(\mathrm{can}^C_{A\Box_H P})^{-1} : (A\Box_H P)\otimes C \to (A\Box_H P)\otimes_{B\otimes D}(A\Box_H P),\\
&(\sum_i a_i\otimes p_i)\otimes c \mapsto \sum_i (a_i c^{[2]}{}_{(-1)}{}^{[1]}\otimes p_i c^{[1]})\otimes_{B\otimes D}(c^{[2]}{}_{(-1)}{}^{[2]}\otimes c^{[2]}{}_{(0)}).
\end{aligned}
\tag{3.14}
$$

Proof. A and P satisfy the assumptions of Lemma 3.2.1, hence $A\Box_H P$ is an algebra and a right C-comodule. Obviously $B\otimes D\subseteq (A\Box_H P)^{\mathrm{co}C}$, and, by Lemma 3.2.3, $A\Box_H P$ is a faithfully flat right $B\otimes D$-module, hence if $\mathrm{can}^C_{A\Box_H P}$ is bijective, then, by Lemma 2.1.1, $(A\Box_H P)^C(B\otimes D)$ is a C-coalgebra Galois extension.

To verify the explicit form of the inverse of the canonical map we first prove that $(\mathrm{can}^C_{A\Box_H P})^{-1}$ is well defined. Observe that the fact that D is a subalgebra of ${}^{\mathrm{co}H}P$ together with the assumption that P is an H-comodule algebra mean that $P\in {}^H_D\mathcal{M}$, hence formula (3.14) makes sense. Denote by

$$
\overline{\overline{(\mathrm{can}^C_{A\Box_H P})^{-1}}} : (A\Box_H P)\otimes C \to (A\otimes P)\otimes_{B\otimes D}(A\otimes P)
$$

the colifting of $(\mathrm{can}^C_{A\Box_H P})^{-1}$, and (for brevity) let $R = B\otimes D$, $Q = A\otimes P$. Since A is a flat left B-module and P is a flat left D-module, Q is a flat left R-module by Lemma 3.2.2. Hence the top row of the following diagram is exact:

$$
\begin{tikzcd}
0 \arrow[r] & (A\Box_H P)\otimes_R Q \arrow[r] & (A\otimes P)\otimes_R Q \arrow[r, shift left, "(\rho^H\otimes P)\otimes Q"] \arrow[r, shift right, "(A\otimes {}^H\rho)\otimes Q"'] & (A\otimes H\otimes P)\otimes_R Q \\
& & (A\Box_H P)\otimes C \arrow[ul, dashed, "\overline{(\mathrm{can}^C_{A\Box_H P})^{-1}}"] \arrow[u, "\overline{\overline{(\mathrm{can}^C_{A\Box_H P})^{-1}}}"'] &
\end{tikzcd}
\tag{3.15}
$$

Hence, in order to prove that there exists the unique factorisation $\overline{(\mathrm{can}^C_{A\Box_H P})^{-1}}$ of $\overline{\overline{(\mathrm{can}^C_{A\Box_H P})^{-1}}}$ (as indicated in the above diagram), it is enough, by the universal property of a kernel, to prove that

$$
((\rho^H\otimes P)\otimes Q)\circ\overline{\overline{(\mathrm{can}^C_{A\Box_H P})^{-1}}} = ((A\otimes {}^H\rho)\otimes Q)\circ\overline{\overline{(\mathrm{can}^C_{A\Box_H P})^{-1}}}.
$$

Take any $\sum_i (a_i \otimes p_i) \otimes c \in (A\Box_H P) \otimes C$ and write,

$$\begin{aligned}
&((A \otimes {}^H\rho) \otimes Q) \circ \overline{\overline{(\mathrm{can}^C_{A\Box_H P})^{-1}}}(\sum_i (a_i \otimes p_i) \otimes c)\\
&\quad = \sum_i (a_i c^{[2]}{}_{(-1)}{}^{[1]} \otimes p_{i(-1)} c^{[1]}{}_{(-1)} \otimes p_{i(0)} c^{[1]}{}_{(0)}) \otimes_R (c^{[2]}{}_{(-1)}{}^{[2]} \otimes c^{[2]}{}_{(0)})\\
&\qquad [\text{ use eq. (3.13) and that } \textstyle\sum_i a_i \otimes p_{i(-1)} \otimes p_{i(0)} = \sum_i a_{i(0)} \otimes a_{i(1)} \otimes p_i \ ]\\
&\quad = \sum_i (a_{i(0)} c^{[2]}{}_{(-1)}{}^{[1]} \otimes a_{i(1)} S c^{\underline{[2]}}{}_{(-2)} \otimes p_i c^{[1]}) \otimes_R (c^{[2]}{}_{(-1)}{}^{[2]} \otimes c^{[2]}{}_{(0)})\\
&\qquad\qquad [\text{ use eq. (2.10) for } c^{[2]}{}_{(-1)} \ ]\\
&\quad = \sum_i (a_{i(0)} c^{[2]}{}_{(-1)}{}^{[1]}{}_{(0)} \otimes a_{i(1)} c^{[2]}{}_{(-1)}{}^{[1]}{}_{(1)} \otimes p_i c^{[1]}) \otimes_R (c^{[2]}{}_{(-1)}{}^{[2]} \otimes c^{[2]}{}_{(0)})\\
&\qquad = ((\rho^H \otimes P) \otimes Q) \circ \overline{\overline{(\mathrm{can}^C_{A\Box_H P})^{-1}}}((\sum_i a_i \otimes p_i) \otimes c).
\end{aligned}$$

Denote $Q' = A\Box_H P$. As A is a right faithfully flat B-module, and P is a right faithfully flat D-module, then, by Lemma 3.2.3, Q' is a right faithfully flat, hence flat, R-module. Therefore the top row in the diagram below is exact:

$$\begin{tikzcd}
0 \arrow[r] & Q' \otimes_R (A\Box_H P) \arrow[r] & Q' \otimes_R (A \otimes P) \arrow[r, shift left, "Q'\otimes(\rho^H \otimes P)"] \arrow[r, shift right, "Q'\otimes(A\otimes {}^H\rho)"'] & Q' \otimes_R (A \otimes H \otimes P). \\
& & (A\Box_H P) \otimes C \arrow[ul, dashed, "(\mathrm{can}^C_{A\Box_H P})^{-1}"] \arrow[u, "\overline{(\mathrm{can}^C_{A\Box_H P})^{-1}}"'] &
\end{tikzcd}$$

Moreover, for any $\sum_i (a_i \otimes p_i) \otimes c \in (A\Box_H P) \otimes C$,

$$\begin{aligned}
&(Q' \otimes (\rho^H \otimes P)) \circ \overline{(\mathrm{can}^C_{A\Box_H P})^{-1}}(\sum_i (a_i \otimes p_i) \otimes c)\\
&\quad = \sum_i (a_i c^{[2]}{}_{(-1)}{}^{[1]} \otimes p_i c^{[1]}) \otimes_R (c^{[2]}{}_{(-1)}{}^{[2]}{}_{(0)} \otimes c^{[2]}{}_{(-1)}{}^{[2]}{}_{(1)} \otimes c^{[2]}{}_{(0)})\\
&\qquad [\text{ use the right } H\text{-colinearity of the translation map for } A^H(B) \text{ (eq. (2.6)) }]\\
&\quad = \sum_i (a_i c^{[2]}{}_{(-2)}{}^{[1]} \otimes c^{[1]}) \otimes_R (c^{[2]}{}_{(-2)}{}^{[2]} \otimes c^{[2]}{}_{(-1)} \otimes c^{[2]}{}_{(0)})\\
&\qquad = (Q' \otimes (A \otimes {}^H\rho)) \circ \overline{(\mathrm{can}^C_{A\Box_H P})^{-1}}(\sum_i (a_i \otimes p_i) \otimes c).
\end{aligned}$$

By the universal property of a kernel, there exists the unique factorization $(\mathrm{can}^C_{A\Box_H P})^{-1}$ of $\overline{(\mathrm{can}^C_{A\Box_H P})^{-1}}$ as required.

It remains to be proven that $(\mathrm{can}^C_{A\Box_H P})^{-1}$ is the inverse of $\mathrm{can}^C_{A\Box_H P}$. For all

$\sum_i (a_i \otimes p_i) \otimes c \in (A\Box_H P) \otimes C$,

$$\begin{aligned}
&\mathrm{can}^C_{A\Box_H P} \circ (\mathrm{can}^C_{A\Box_H P})^{-1}(\sum_i (a_i \otimes p_i) \otimes c) = \\
&\qquad = \sum_i a_i c^{[2]}{}_{(-1)}{}^{[1]} c^{[2]}{}_{(-1)}{}^{[2]} \otimes p_i c^{[1]} c^{[2]}{}_{(0)} \otimes c^{[2]}{}_{(1)} \\
&[\text{use } (2.7)] = \sum_i a_i \varepsilon^H (c^{[2]}{}_{(-1)}) \otimes p_i c^{[1]} c^{[2]}{}_{(0)} \otimes c^{[2]}{}_{(1)} \\
&\qquad = \sum_i a_i \otimes p_i c^{[1]} c^{[2]}{}_{(0)} \otimes c^{[2]}{}_{(1)} = \sum_i (a_i \otimes p_i) \otimes c.
\end{aligned}$$

On the other hand, for any $\sum_{ij} (a_i \otimes p_i) \otimes_R (a'_j \otimes p'_j) \in (A\Box_H P) \otimes_R (A\Box_H P)$,

$$\begin{aligned}
&(\mathrm{can}^C_{A\Box_H P})^{-1} \circ \mathrm{can}^C_{A\Box_H P}(\sum_{ij} (a_i \otimes p_i) \otimes_R (a'_j \otimes p'_j)) \\
&= \sum_{ij} (a_i a'_j p'_{j(1)}{}^{[2]}{}_{(-1)}{}^{[1]} \otimes p_i p'_{j(0)} p'_{j(1)}{}^{[1]}) \otimes_R (p'_{j(1)}{}^{[2]}{}_{(-1)}{}^{[2]} \otimes p'_{j(1)}{}^{[2]}{}_{(0)}) \\
&[\text{ use } (2.8)\ ] = \sum_{ij} (a_i a'_j p'_{j(-1)}{}^{[1]} \otimes p_i) \otimes_R (p'_{j(-1)}{}^{[2]} \otimes p'_{j(0)}) \\
&[\text{use that } \textstyle\sum_j a'_{j(0)} \otimes a'_{j(1)} \otimes p'_j = \sum_j a'_j \otimes p'_{j(-1)} \otimes p'_{j(0)}\ ] \\
&= \sum_{ij} (a_i a'_{j(0)} a'_{j(1)}{}^{[1]} \otimes p_i) \otimes_R (a'_{j(1)}{}^{[2]} \otimes p'_j) \\
&[\text{ use } (2.8)\ ] = \sum_{ij} (a_i \otimes p_i) \otimes_R (a'_j \otimes p'_j).
\end{aligned}$$

Thus, we conclude that $(\mathrm{can}^C_{A\Box_H P})^{-1}$ is the inverse of $\mathrm{can}^C_{A\Box_H P}$ as stated. This completes the proof of the theorem. □

The assumption about the faithful flatness of A and P, although quite restrictive, is a usual assumption made for Galois-type extensions to be able to view them as bona fide generalisations of torsors or principal bundles. Indeed, if one wants to develop a differential geometry on Galois-type extensions in terms of strong connections, the faithful flatness becomes necessary (cf. Theorem 2.5 [7]).

Example 3.2.6. ([38] Remark 3.11 (2)) Let A be an H-Hopf Galois extension of B, faithfully flat as a right B-module, and let P be a Hopf algebra faithfully flat as a $\mathbb{K}$-module. Suppose that $f : P \to H$ is a Hopf algebra morphism. Define a left H-coaction on P by $p \mapsto f(p_{(1)}) \otimes p_{(2)}$. As $D = P^{\mathrm{co}P} = \mathbb{K}1_P$ and $\mathbb{K}1_P \subseteq {}^{\mathrm{co}H}P$, all the assumptions of Theorem 3.2.5 are satisfied, and so $A\Box_H P$ is a P-Hopf Galois extension of $B \otimes \mathbb{K}1_P \simeq B$. This P-Hopf Galois extension is called a P-prolongation of A if f is surjective.

3.3 Example: The cotensor product of Matsumoto spheres I

The coordinate algebra of the *Matsumoto sphere* ([30],[31]), denoted $\vartheta(S^3_\theta)$, is the $*$-algebra over $\mathbb{C}$ generated by the elements a and b which satisfy the relations

$$aa^* = a^*a,\ \ bb^* = b^*b,\ \ ab = \lambda ba,\ \ ab^* = \bar{\lambda} b^*a,\ \ aa^* + bb^* = 1, \tag{3.16}$$

where $\lambda = e^{2\pi i\theta}$, $\theta \in \mathbb{R}$. Note that there exists a $\mathbb{Z}$-grading on $\vartheta(S^3_\theta)$ defined on generators by

$$\deg(a) = 1,\ \deg(a^*) = -1,\ \deg(b) = 1,\ \deg(b^*) = -1, \tag{3.17}$$

which agrees with the $*$-operation in the sense that, for all homogeneous $p \in \vartheta(S^3_\theta)$, $\deg(p^*) = -\deg(p)$.

Denote by $\vartheta(U(1))$ the coordinate $*$-Hopf algebra of $U(1)$, generated by the unitary and grouplike element u, i.e.,

$$uu^* = u^*u = 1,\ \ \Delta(u) = u \otimes u,\ \ Su = u^*,\ \ \varepsilon(u) = 1. \tag{3.18}$$

The $\mathbb{Z}$-grading (3.17) naturally makes $\vartheta(S^3_\theta)$ a $\vartheta(U(1))$-comodule algebra by defining a coaction

$$\rho^{\vartheta(U(1))}(p) = p \otimes u^{\deg(x)},\ \text{for all homogeneous } p \in \vartheta(S^3_\theta). \tag{3.19}$$

It is easy to see that the subalgebra of coinvariants of this coaction, $\vartheta(S^3_\theta)^{\mathrm{co}\vartheta(U(1))} = \{x \in \vartheta(S^3_\theta) | \deg(x) = 0\}$, is a commutative $*$-algebra generated by the elements $z = aa^*$, $x_+ = ba^*$, $x_- = (x_+)^* = ab^*$, which satisfy the relation

$$z^2 + x_+x_- = 1, \tag{3.20}$$

and hence it can be identified with the coordinate algebra $\vartheta(S^2)$ of the classical two-sphere.

Let us adopt the notational convention $a^{-n} \equiv (a^*)^n$, $b^{-n} \equiv (b^*)^n$, $(x_+)^{-n} \equiv (x_-)^n$, for all $n \in \mathbb{N}$. It is obvious that the monomials

$$(aa^*)^n a^\mu b^\nu,\ \ n \in \mathbb{N}_0,\ \mu, \nu \in \mathbb{Z}, \tag{3.21}$$

span $\vartheta(S^3_\theta)$ as a vector space. Moreover, if $\mu\nu < 0$, then

$$\begin{aligned}(aa^*)^n a^\mu b^\nu &= \lambda^{\frac{\nu(\nu-1)}{2} + \mu\nu} z^n (x_+)^\nu a^{\mu+\nu} \text{ if } |\nu| \le |\mu|,\\ (aa^*)^n a^\mu b^\nu &= \lambda^{-\frac{\mu(\mu-1)}{2}} z^n (x_+)^{-\mu} b^{\mu+\nu} \text{ if } |\nu| > |\mu|.\end{aligned} \tag{3.22}$$

It can be shown ([12]) that $\vartheta(S^3_\theta)(\vartheta(S^2))^{\vartheta(U(1))}$ is a principal $\vartheta(U(1))$-Hopf Galois extension with the inverse of the canonical map given, for any $p \in \vartheta(S^3_\theta)$ and $n \in \mathbb{N}_0$, by

$$\begin{aligned}\mathrm{can}^{\vartheta(U(1))}_{\vartheta(S^3_\theta)}(p \otimes u^n) &= \sum_{m=0}^{n} \binom{n}{m} p b^{*m} a^{*n-m} \otimes_B a^{n-m} b^m,\\ \mathrm{can}^{\vartheta(U(1))}_{\vartheta(S^3_\theta)}(p \otimes u^{*n}) &= \sum_{m=0}^{n} \binom{n}{m} p b^{m} a^{n-m} \otimes_B a^{*n-m} b^{*m}.\end{aligned} \tag{3.23}$$

Consider two copies of the Matsumoto sphere S^3_θ and $S^3_{\theta'}$ with not necessarily equal deformation parameters. To relieve the notation, we denote $A = \vartheta(S^3_\theta)$, $P = \vartheta(S^3_{\theta'})$, $H = \vartheta(U(1))$, $B = A^{\mathrm{co}H}$, $D = P^{\mathrm{co}H}$. Furthermore, for all $n \in \mathbb{Z}$, let A_n (resp. P_n) denote the grade n part of A (resp. P). Moreover, we put the apostrophe at the elements of the coordinate algebra of the Matsumoto sphere $P = \vartheta(S^3_{\theta'})$, to distinguish them from the respective elements of $A = \vartheta(S^3_\theta)$, i.e., the elements a, b are the generators of A which satisfy the relations (3.16), and the elements a' and b' are the generators of P which satisfy the relations

$$a'a'^* = a'^*a', \;\; b'b'^* = b'^*b', \;\; a'b' = \lambda'b'a', \;\; a'b'^* = \bar{\lambda}'b'^*a', \;\; a'a'^* + b'b'^* = 1, \quad (3.24)$$

where $\lambda' = e^{2\pi i\theta'}$. Similarily, we denote $z' = a'^*a'$, $x'_+ = b'a'^*$, $x'_- = a'b'^*$.

Define the left $*$-comodule algebra H-coaction on P by

$$^H\rho(p) = u^{-\deg(p)} \otimes p \text{ for all homogeneous elements } p \in P. \quad (3.25)$$

Note that $^{\mathrm{co}H}P = B = P^{\mathrm{co}H}$.

Obviously, the cotensor product

$$A\Box_H P = \bigoplus_{n\in\mathbb{Z}} A_n \otimes P_{-n},$$

and hence it is generated as a vector space by the elements of the form (cf. (3.21)),

$$(aa^*)^m a^\alpha b^\beta \otimes (a'a'^*)^n a'^\mu b'^\nu, \;\; m,n \in \mathbb{N}_0, \; \alpha,\beta,\mu,\nu \in \mathbb{Z}, \; \alpha+\beta+\mu+\nu = 0. \quad (3.26)$$

Denote

$$\alpha = a \otimes a'^*, \;\; \beta = b \otimes b'^*, \;\; \gamma = a \otimes b'^*, \;\; \delta = b \otimes a'^*, \quad (3.27)$$

It is easy to see using (3.16) and (3.24) that

$$\begin{array}{lll} z \otimes 1 = \alpha^*\alpha + \gamma^*\gamma, & x_+ \otimes 1 = \delta\alpha^* + \beta\gamma^*, & x_- \otimes 1 = \alpha\delta^* + \gamma\beta^*, \\ 1 \otimes z' = \alpha^*\alpha + \delta^*\delta, & 1 \otimes x'_+ = \gamma^*\alpha + \beta^*\delta, & 1 \otimes x'_- = \alpha^*\gamma + \delta^*\beta. \end{array} \quad (3.28)$$

Therefore, using (3.22), it is clear that α, β, γ, δ generate $A\Box_H P$ as a $*$-algebra.

One easily checks that α, β, γ, δ satisfy the following commutation relations

$$\begin{array}{llll} \alpha\alpha^* = \alpha^*\alpha, & \beta\beta^* = \beta^*\beta, & \gamma\gamma^* = \gamma^*\gamma, & \delta\delta^* = \delta^*\delta, \\ \alpha\beta = \lambda\lambda'\beta\alpha, & \alpha\beta^* = \bar{\lambda}\bar{\lambda}'\beta^*\alpha, & \alpha\gamma = \lambda'\gamma\alpha, & \alpha\gamma^* = \bar{\lambda}'\gamma^*\alpha, \\ \alpha\delta = \lambda\delta\alpha, & \alpha\delta^* = \bar{\lambda}\delta^*\alpha, & \beta\gamma = \bar{\lambda}\gamma\beta, & \beta\gamma^* = \lambda\gamma^*\beta, \\ \beta\delta = \bar{\lambda}'\delta\beta, & \beta\delta^* = \lambda'\delta^*\beta, & \gamma\delta = \lambda\bar{\lambda}'\delta\gamma, & \gamma\delta^* = \bar{\lambda}\lambda'\delta^*\gamma, \end{array} \quad (3.29)$$

and, in addition,

$$\begin{array}{c} \alpha^*\alpha + \beta^*\beta + \gamma^*\gamma + \delta^*\delta = 1, \\ \alpha\beta = \lambda'\gamma\delta. \end{array} \quad (3.30)$$

Let $\bar{A}$ (resp. $\bar{P}$) be the free $*$-algebra generated by the elements a and b (resp. a', b'). Let us identify a with $a \otimes 1$, a' with $1 \otimes a'$, etc. Then $A \otimes P$ is isomorphic to $\bar{A} \otimes \bar{P}/I$ where I is the ideal in $\bar{A} \otimes \bar{P}$ generated by the relations (3.16) and (3.24). Accordingly, $A\Box_H P$ is isomorphic to $\bar{A}\Box_H\bar{P}/(I \cap \bar{A}\Box_H\bar{P})$. Argument based on the grading allows for easy identification of $I \cap \bar{A}\Box_H\bar{P}$, and then the tedious but simple check proves that $I \cap \bar{A}\Box_H\bar{P} \subseteq J$, where J is the ideal in $\bar{A}\Box_H\bar{P}$ generated by relations (3.29) and (3.30). Hence $A\Box_H P$ is isomorphic to the free $*$-algebra generated by $\alpha, \beta, \gamma, \delta$ modulo the relations (3.29) and (3.30).

By Theorem 2.2.6 and the existence of a strong connection on the Matsumoto sphere ([12]), A and P are faithfully flat as right and left B-modules. Therefore, by Theorem 3.2.5, $(A\Box_H P)^H(B \otimes D)$ is an H-Hopf Galois extension. Let $n \in \mathbb{N}$. Substituting expressions (3.23) for translation maps of A and P in the formula (3.14), yields

$$\begin{aligned}
\tau^H_{A\Box_H P}(u^n) &= \sum_{p,m=0}^{n} \binom{n}{p}\binom{n}{m} (b^p a^{n-p} \otimes (b'^*)^m (a'^*)^{n-m}) \\
&\qquad \otimes_{B\otimes D} ((a^*)^{n-p}(b^*)^p \otimes a'^{n-m} b'^m), \\
\tau^H_{A\Box_H P}(u^{-n}) &= \sum_{p,m=0}^{n} \binom{n}{p}\binom{n}{m} ((b^*)^p (a^*)^{n-p} \otimes b'^m a'^{n-m}) \\
&\qquad \otimes_{B\otimes D} (a^{n-p} b^p \otimes (a'^*)^{n-m} (b'^*)^m).
\end{aligned}$$

Finally, expressing the above results in terms of generators α, β, γ gives

$$\begin{aligned}
\tau^H_{A\Box_H P}(u^n) &= \sum_{p=0}^{n}\sum_{m=0}^{p-1} \binom{n}{p}\binom{n}{m} \alpha^{n-p}\delta^{p-m}\beta^m \otimes_{B\otimes B} \beta^{*m}\delta^{*p-m}\alpha^{*n-p} \\
&\quad + \sum_{p=0}^{n}\sum_{m=p}^{n} \binom{n}{p}\binom{n}{m} \alpha^{n-m}\gamma^{m-p}\beta^p \otimes_{B\otimes B} \beta^{*p}\gamma^{*m-p}\alpha^{*n-m},
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
\tau^H_{A\Box_H P}(u^{-n}) &= \sum_{p=0}^{n}\sum_{m=0}^{p-1} \binom{n}{p}\binom{n}{m} \alpha^{*n-p}\delta^{*p-m}\beta^{*m} \otimes_{B\otimes B} \beta^m\delta^{p-m}\alpha^{n-p} \\
&\quad + \sum_{p=0}^{n}\sum_{m=p}^{n} \binom{n}{p}\binom{n}{m} \alpha^{*n-m}\gamma^{*m-p}\beta^{*p} \otimes_{B\otimes B} \beta^p\gamma^{m-p}\alpha^{n-m}.
\end{aligned} \tag{3.32}$$

3.4 Strong connections and cotensor products

The previous section was concerned with the cotensor product $A\Box_H P$ of an H-Hopf Galois extension A and a C-coalgebra Galois extension P such that P is an (H, C)-bicomodule and $P^{\mathrm{co}C} \subseteq {}^{\mathrm{co}H}P$. In the present section we drop the latter assumption. The formula for the inverse of the canonical map (3.14) becomes now badly defined, because the left coaction of H on P no longer commutes with the multiplication by elements of $P^{\mathrm{co}C}$. Hence we need a quantity associated with the inverse of the canonical map, but with image in the tensor product over the ground ring. The obvious candidate is the strong connection form (Definition 2.2.3). However, in order to formulate our results as generally as possible, we introduce a more general notion.

Definition 3.4.1. Let C be a coalgebra and P be an algebra and a right C-comodule. Let $\tilde{\mathrm{can}}_P^C : P \otimes P \to P \otimes C$, $p \otimes p' \to pp'_{(0)} \otimes p'_{(1)}$ be a lifting of the canonical map. A linear morphism $\tilde{\tau}_P^C : C \to P \otimes P$, such that, for all $c \in C$, $\tilde{\mathrm{can}}_P^C(\tilde{\tau}_P^C(c)) = 1_P \otimes c$, is called *a colifting of the translation map*. For convenience in computations we use a 'Sweedler-like' notation, $\tilde{\tau}_P^C(c) = c^{\underline{[1]}} \otimes c^{\underline{[2]}}$ (summation understood), similar to that for a strong connection form (eq. (2.21)).

The following lemma is the core of the results of this section.

Lemma 3.4.2. *Let H be a Hopf algebra and C be a coalgebra flat as a $\mathbb{K}$-module. Let A be a right H-comodule algebra, and P be an (H, C)-bicomodule such that the left H-coaction is an algebra map. Assume that $A \otimes P$ and $A\Box_H P$ are flat $\mathbb{K}$-modules. Suppose that the following conditions are satisfied.*

1. *There exists on A a colifting $\tilde{\tau}_A^H : H \to A \otimes A$, $h \mapsto h^{\underline{[1]}} \otimes h^{\underline{[2]}}$ of the translation map, such that, for any $h \in H$,*

$$(A \otimes \rho^H) \circ \tilde{\tau}_A^H(h) = \tilde{\tau}_A^H(h_{(1)}) \otimes h_{(2)} \text{ (right covariance)}, \tag{3.33}$$

$$h^{\underline{[1]}}{}_{(1)} \otimes h^{\underline{[1]}}{}_{(0)} \otimes h^{\underline{[2]}} = Sh_{(1)} \otimes h_{(2)}{}^{\underline{[1]}} \otimes h_{(2)}{}^{\underline{[2]}} \text{ (left covariance)}. \tag{3.34}$$

2. *There exists on P a colifting $\tilde{\tau}_P^C : C \to P \otimes P$, $c \mapsto c^{\underline{[1]}} \otimes c^{\underline{[2]}}$ of the translation map, such that, for any $c \in C$,*

$$c^{\underline{[1]}}{}_{(-1)} \otimes c^{\underline{[1]}}{}_{(0)} \otimes c^{\underline{[2]}} = Sc^{\underline{[2]}}{}_{(-1)} \otimes c^{\underline{[1]}} \otimes c^{\underline{[2]}}{}_{(0)}. \tag{3.35}$$

Then $A\Box_H P$ is an algebra and a right C-comodule and the map

$$\begin{aligned} &\tilde{\tau}_{A\Box_H P}^C : C \to (A\Box_H P) \otimes (A\Box_H P), \\ &c \mapsto (c^{\underline{[2]}}{}_{(-1)}{}^{\underline{[1]}} \otimes c^{\underline{[1]}}) \otimes (c^{\underline{[2]}}{}_{(-1)}{}^{\underline{[2]}} \otimes c^{\underline{[2]}}{}_{(0)}), \end{aligned} \tag{3.36}$$

is a colifting of the translation map on $A\Box_H P$, i.e.,

$$\tilde{\mathrm{can}}_{A\Box_H P}^C \circ \tilde{\tau}_{A\Box_H P}^C(c) = (1_A \otimes 1_P) \otimes c.$$

Proof. $A\Box_H P$ is an algebra and a right C-comodule by Lemma 3.2.1. Since, by assumption, $A \otimes P$ and $A\Box_H P$ are flat as $\mathbb{K}$-modules, in order to prove that the map $\tilde{\tau}_{A\Box_H P}^C$, defined by (3.36), has its image in $(A\Box_H P) \otimes (A\Box_H P)$, it is enough to show that

$$((\rho^H \otimes P) \otimes (A \otimes P)) \circ \tilde{\tau}_{A\Box_H P}^C = ((A \otimes {}^H\rho) \otimes (A \otimes P)) \circ \tilde{\tau}_{A\Box_H P}^C$$

and

$$((A \otimes P) \otimes (\rho^H \otimes P)) \circ \tilde{\tau}_{A\Box_H P}^C = ((A \otimes P) \otimes (A \otimes {}^H\rho)) \circ \tilde{\tau}_{A\Box_H P}^C$$

(cf. the proof of Theorem 3.2.5).

Take any $c \in C$ and compute

$$
\begin{aligned}
&((\rho^H \otimes P) \otimes (A \otimes P)) \circ \tilde{\tau}^C_{A\Box_H P}(c) \\
&\qquad = ({c^{\underline{[2]}}}_{(-1)}{}^{\underline{[1]}}{}_{(0)} \otimes {c^{\underline{[2]}}}_{(-1)}{}^{\underline{[1]}}{}_{(1)} \otimes c^{\underline{[1]}}) \otimes ({c^{\underline{[2]}}}_{(-1)}{}^{\underline{[2]}} \otimes {c^{\underline{[2]}}}_{(0)}) \\
[\text{use } (3.34)] &\qquad = ({c^{\underline{[2]}}}_{(-1)(2)}{}^{\underline{[1]}} \otimes S{c^{\underline{[2]}}}_{(-1)(1)} \otimes c^{\underline{[1]}}) \otimes ({c^{\underline{[2]}}}_{(-1)(2)}{}^{\underline{[2]}} \otimes {c^{\underline{[2]}}}_{(0)}) \\
&\qquad = ({c^{\underline{[2]}}}_{(0)(-1)}{}^{\underline{[1]}} \otimes S{c^{\underline{[2]}}}_{(-1)} \otimes c^{\underline{[1]}}) \otimes ({c^{\underline{[2]}}}_{(0)(-1)}{}^{\underline{[2]}} \otimes {c^{\underline{[2]}}}_{(0)(0)}) \\
[\text{ use } 3.35\ ] &\qquad = ({c^{\underline{[2]}}}_{(-1)}{}^{\underline{[1]}} \otimes {c^{\underline{[1]}}}_{(-1)} \otimes {c^{\underline{[1]}}}_{(0)}) \otimes ({c^{\underline{[2]}}}_{(-1)}{}^{\underline{[2]}} \otimes {c^{\underline{[2]}}}_{(0)}) \\
&\qquad = ((A \otimes {}^H\rho) \otimes (A \otimes P)) \circ \tilde{\tau}^C_{A\Box_H P}(c).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&((A \otimes P) \otimes (\rho^H \otimes P)) \circ \tilde{\tau}^C_{A\Box_H P}(c) \\
&\qquad = ({c^{\underline{[2]}}}_{(-1)}{}^{\underline{[1]}} \otimes c^{\underline{[1]}}) \otimes ({c^{\underline{[2]}}}_{(-1)}{}^{\underline{[2]}}{}_{(0)} \otimes {c^{\underline{[2]}}}_{(-1)}{}^{\underline{[2]}}{}_{(1)} \otimes {c^{\underline{[2]}}}_{(0)}) \\
[\text{ use } (3.33)\ ] &\qquad = ({c^{\underline{[2]}}}_{(-2)}{}^{\underline{[1]}} \otimes c^{\underline{[1]}}) \otimes ({c^{\underline{[2]}}}_{(-2)}{}^{\underline{[2]}} \otimes {c^{\underline{[2]}}}_{(-1)} \otimes {c^{\underline{[2]}}}_{(0)}) \\
&\qquad = ((A \otimes P) \otimes (A \otimes {}^H\rho)) \circ \tilde{\tau}^C_{A\Box_H P}(c).
\end{aligned}
$$

Finally we check whether (3.36) is a colifting of the translation map for $A\Box_H P$. Note that $m \circ \tilde{\tau}^H_A(c) = \varepsilon(c)$, for any $c \in C$, hence

$$
\begin{aligned}
\tilde{\mathrm{can}}^C_{A\Box_H P} \circ \tilde{\tau}^C_{A\Box_H P}(c) &= {c^{\underline{[2]}}}_{(-1)}{}^{\underline{[1]}} {c^{\underline{[2]}}}_{(-1)}{}^{\underline{[2]}} \otimes c^{\underline{[1]}} {c^{\underline{[2]}}}_{(0)(0)} \otimes {c^{\underline{[2]}}}_{(0)(1)} \\
&= \varepsilon({c^{\underline{[2]}}}_{(-1)}) 1_A \otimes c^{\underline{[1]}} {c^{\underline{[2]}}}_{(0)(0)} \otimes {c^{\underline{[2]}}}_{(0)(1)} \\
&= (1_A \otimes 1_P) \otimes c.
\end{aligned}
$$

This completes the proof of the lemma. □

We want to study the rules governing the existence of a strong connection on $A\Box_H P$, depending on the existence and properties of strong connection forms on A and P. However, for the definition of a strong connection form to make sense, we first need an entwining.

Let $(P, C)_\psi$ be an entwining structure. Assume that P is a left H-comodule for a coalgebra H. We say that entwining ψ (and its inverse if it exists) *commutes with the left H-coaction* if ψ is a left H-colinear map, where we view $C \otimes P$ and $P \otimes C$ as left H-comodules via the left H-coaction on the P-leg of the respective tensor products.

Lemma 3.4.3. *Let H be a bialgebra, and $(P, C)_\psi$ be an entwining structure such that P is an entwined module. Suppose that P is a left H-comodule algebra such that ψ commutes with the left H-coaction. Then P is an (H, C)-bicomodule if and only if*

$$({}^H\rho \otimes C) \circ \rho^C(1_P) = 1_H \otimes \rho^C(1_P). \tag{3.37}$$

Proof. Since $P \in \mathcal{M}_P^C(\psi)$, the right coaction $\rho^C : P \to P \otimes C$ is necessarily given by $p \mapsto 1_{(0)}\psi(1_{(1)} \otimes p)$. Hence, for all $p \in P$,

$$
\begin{aligned}
({}^H\rho \otimes C) \circ \rho^C(p) & \\
&= {}^H\rho(1_{(0)}p_\alpha) \otimes 1_{(1)}{}^\alpha && (\text{since } P \in \mathcal{M}_P^C(\psi)) \\
&= {}^H\rho(1_{(0)}){}^H\rho(p_\alpha) \otimes 1_{(1)}{}^\alpha && (\text{since } {}^H\rho \text{ is algebraic}) \\
&= 1_H p_{\alpha(-1)} \otimes 1_{(0)}p_{\alpha(0)} \otimes 1_{(1)}{}^\alpha && (\text{by } (3.37)) \\
&= p_{(-1)} \otimes 1_{(0)}p_{(0)\alpha} \otimes 1_{(1)}{}^\alpha && (\text{since } \psi \text{ commutes with } H\text{-coaction}) \\
&= (H \otimes \rho^C) \circ {}^H\rho(p) && (\text{since } P \in \mathcal{M}_P^C(\psi)).
\end{aligned}
$$

Obviously, if P is an (H,C)-bicomodule, then (3.37) is satisfied, since ${}^H\rho(1_P) = 1_H \otimes 1_P$. □

Note that condition (3.37) is automatically satisfied if there exists a grouplike element $e \in C$ such that, $\rho^C(1) = 1 \otimes e$ (thus, in particular, for e-copointed C-coalgebra Galois extensions).

Lemma 3.4.4. *Let $P(B,C,\psi)$ be a $(P,C)_\psi$-extension and let H be a Hopf algebra. Let P be an (H,C)-bicomodule and an H-comodule algebra. Suppose that either:*

(a) *there exists a colifting $\tilde{\tau}_P^C : c \mapsto c^{\underline{[1]}} \otimes c^{\underline{[2]}}$ of the translation map such that, for any $c \in C$,*

$$c^{\underline{[1]}}{}_{(-1)}c^{\underline{[2]}}{}_{(-1)} \otimes c^{\underline{[1]}}{}_{(0)} \otimes c^{\underline{[2]}}{}_{(0)} = 1_H \otimes c^{\underline{[1]}} \otimes c^{\underline{[2]}}, \tag{3.38}$$

or

(b) *$P(B)^C$ is a C-coalgebra Galois extension and $B \subseteq {}^{coH}P$.*

Then the entwining ψ commutes with the left H-coaction ${}^H\rho : P \to H \otimes P$.

Proof. Assume that there exists a colifting satisfying condition (3.38). Since P is an (H,C)-bicomodule,

$$p_{(-1)} \otimes 1_{P(0)}\psi(1_{P(1)} \otimes p_{(0)}) = p_{\alpha(-1)} \otimes 1_{P(0)}p_{\alpha(0)} \otimes 1_{P(1)}{}^\alpha, \tag{3.39}$$

for any $p \in P$ and $c \in C$. By the definition of a colifting, $c^{\underline{[1]}}c^{\underline{[2]}}{}_{(0)} \otimes c^{\underline{[2]}}{}_{(1)} = 1_P \otimes c$, hence we can write,

$$\psi(c \otimes p) = c^{\underline{[1]}}c^{\underline{[2]}}{}_{(0)}\psi(c^{\underline{[2]}}{}_{(1)} \otimes p) = c^{\underline{[1]}}\rho^C(c^{\underline{[2]}}p) = c^{\underline{[1]}}1_{P(0)}\psi(1_{P(1)} \otimes c^{\underline{[2]}}p),$$

where the second and third equalities are consequences of the fact that P is an entwined module. Take any $c \in C$, $p \in P$, and compute

$$
\begin{aligned}
p_{(-1)} \otimes \psi(c \otimes p_{(0)}) &= p_{(-1)} \otimes c^{\underline{[1]}} 1_{P(0)} \psi(1_{P(1)} \otimes c^{\underline{[2]}} p_{(0)}) \\
[\text{ use (3.38) }] &= c^{\underline{[1]}}{}_{(-1)} c^{\underline{[2]}}{}_{(-1)} p_{(-1)} \otimes c^{\underline{[1]}}{}_{(0)} 1_{P(0)} \psi(1_{P(1)} \otimes c^{\underline{[2]}}{}_{(0)} p_{(0)}) \\
&= c^{\underline{[1]}}{}_{(-1)} (c^{\underline{[2]}} p)_{(-1)} \otimes c^{\underline{[1]}}{}_{(0)} 1_{P(0)} \psi(1_{P(1)} \otimes (c^{\underline{[2]}} p)_{(0)}) \\
[\text{use (3.39)}] &= c^{\underline{[1]}}{}_{(-1)} (c^{\underline{[2]}} p)_{\alpha(-1)} \otimes c^{\underline{[1]}}{}_{(0)} 1_{P(0)} (c^{\underline{[2]}} p)_{\alpha(0)} \otimes 1_{P(1)}{}^{\alpha} \\
[\ {}^{H}\rho \text{ is algebraic, hence } 1_H \otimes 1_{P(0)} \otimes 1_{P(1)} &= 1_{P(-1)} \otimes 1_{P(0)} \otimes 1_{P(1)} \ ] \\
&= (c^{\underline{[1]}} 1_{P(0)} c^{\underline{[2]}}{}_{\alpha} p_{\beta})_{(-1)} \otimes (c^{\underline{[1]}} 1_{P(0)} c^{\underline{[2]}}{}_{\alpha} p_{\beta})_{(0)} \otimes 1_{P(1)}{}^{\alpha\beta} \\
[\text{ notice that } c^{\underline{[1]}} 1_{P(0)} c^{\underline{[2]}}{}_{\alpha} \otimes 1_{P(1)}{}^{\alpha} &= c^{\underline{[1]}} \rho^{C}(c^{\underline{[2]}}) = 1 \otimes c \ ] \\
&= p_{\beta(-1)} \otimes p_{\beta(0)} \otimes c^{\beta}.
\end{aligned}
$$

Note the similarity of the condition (3.38) to the property (3.12) of the translation map. Indeed, (3.12) has the same form as (3.38), but with the adorned tensor products. Thus a similar argument to the one above, but with the translation map instead of the colifting of the translation map can be used to show that the hypothesis **(b)** implies the assertion. □

Conditions (3.38) or $B \subseteq {}^{coH}P$ are not necessary. For instance, if C is a Hopf algebra and $P(B)^C$ is a C-Hopf Galois extension, then, for the canonical entwining associated to $P(B)^C$, $\psi_{\text{can}}(c \otimes p) = \text{can}_P^C((\text{can}_P^C)^{-1}(1 \otimes c)p) = p_{(0)} \otimes p_{(1)}c$, to commute with the left H-coaction, it is enough that P is an (H, C)-bicomodule.

The following lemma is concerned with the existence of an entwining structure on the cotensor product $A\square_H P$, induced by the entwining structure on P.

Lemma 3.4.5. *Suppose that H is a bialgebra and C is a coalgebra flat as a $\mathbb{K}$-module. Let A, P be, respectively, right and left H-comodule algebras. Let $(P, C)_\psi$ be an entwining structure such that ψ commutes with the left H-coaction. Then the following statements are true.*

1. *$(A\square_H P)_{\psi_\square}$ is an entwining structure with*

$$
\begin{aligned}
\psi_\square : C \otimes (A\square_H P) &\to (A\square_H P) \otimes C, \\
c \otimes (\sum_i a_i \otimes p_i) &\mapsto \sum_i (a_i \otimes p_{i\alpha}) \otimes c^{\alpha}.
\end{aligned} \tag{3.40}
$$

2. *If ψ is invertible then $\psi_\square$ is also an invertible map. Explicitly*

$$
\begin{aligned}
\psi_\square{}^{-1} : (A\square_H P) \otimes C &\to C \otimes (A\square_H P), \\
(\sum_i a_i \otimes p_i) \otimes c &\mapsto c_A \otimes (\sum_i a_i \otimes p_i{}^{A}).
\end{aligned} \tag{3.41}
$$

3. *If $P \in \mathcal{M}_P^C(\psi)$ and $({}^{H}\rho \otimes C) \circ \rho^C(1_P) = 1_H \otimes \rho^C(1_P)$, then $A\square_H P \in \mathcal{M}^C_{A\square_H P}(\psi_\square)$.*

4. *If P is e-copointed then $A\Box_H P$ is e-copointed.*

Proof. As C is a flat $\mathbb{K}$-module, in order to prove that $\psi_\Box$ has its image in $(A\Box_H P)\otimes C$, it is enough to show that $(\rho^H\otimes P\otimes C)\circ\psi_\Box = (A\otimes {}^H\rho\otimes C)\circ\psi_\Box$. For any $c\otimes(\sum_i a_i\otimes p_i)\in C\otimes(A\Box_H P)$:

$$\begin{aligned}(\rho^H\otimes P\otimes C)\circ\psi_\Box(c\otimes(\sum_i a_i\otimes p_i)) &= \sum_i(a_{i(0)}\otimes a_{i(1)}\otimes p_{i\alpha})\otimes c^\alpha\\ &= \sum_i(a_i\otimes p_{i(-1)}\otimes p_{i(0)\alpha})\otimes c^\alpha = \sum_i(a_i\otimes p_{i\alpha(-1)}\otimes p_{i\alpha(0)})\otimes c^\alpha\\ &= (A\otimes {}^H\rho\otimes C)\psi_\Box(c\otimes(\sum_i a_i\otimes p_i)).\end{aligned}$$

That $\psi_\Box$ is an entwining follows easily from the definition of $\psi_\Box$. Moreover, if ψ is invertible, ψ^{-1} also commutes with the left H-coaction, and one can prove, with the similar computations as those for $\psi_\Box$, that the map $\psi_\Box^{-1}$ has its image in $A\Box_H P$. Obviously $\psi_\Box^{-1}$ is the inverse of $\psi_\Box$.

Assume that $P\in\mathcal{M}_P^C(\psi)$ and $({}^H\rho\otimes C)\circ\rho^C(1_P) = 1_H\otimes\rho^C(1_P)$. Then, by Lemma 3.4.3, P is an (H,C)-bicomodule, hence $A\Box_H P$ is an algebra and a C-comodule (by Lemma 3.2.1). Moreover, for any $\sum_i a_i\otimes p_i, \sum_j a'_j\otimes p'_j\in A\Box_H P$,

$$\begin{aligned}\rho^C((\sum_i a_i\otimes p_i)(\sum_j a'_j\otimes p'_j)) &= \sum_{ij} a_ia'_j\otimes\rho^C(p_ip'_j)\\ &= \sum_{ij} a_ia'_j\otimes p_{i(0)}\psi(p_{i(1)}\otimes p'_j)\\ &= (\sum_i a_i\otimes p_i)_{(0)}\psi_\Box((\sum_i a_i\otimes p_i)_{(1)}\otimes(\sum_j a'_j\otimes p'_j)),\end{aligned}$$

hence $A\Box_H P\in\mathcal{M}^C_{A\Box_H P}(\psi_\Box)$. If P is e-copointed, then

$$\rho^C(1_A\otimes 1_P) = 1_A\otimes\rho^C(1_P) = (1_A\otimes 1_P)\otimes e = 1_{A\Box_H P}\otimes e,$$

hence $A\Box_H P$ is e-copointed. □

In what follows we shall need this simple corollary to Theorem 2.2.7.

Corollary 3.4.6. *If $\mathbb{K}$ is a field, C is a coseparable coalgebra, $P_e(B,C,\psi)$ is an e-copointed $(P,C)_\psi$-extension with bijective ψ, and there exists a colifting $\tilde{\tau}_P^C : C\to P\otimes P$ of the translation map, then $\mathrm{can}_P^C : P\otimes_B P\to P\otimes C$ is bijective and $P(B)_e^C$ is a principal C-coalgebra Galois extension.*

Proof. Define the map

$$f : P\otimes C\ni p\otimes c\mapsto p\tilde{\tau}_P^C(c)\in P\otimes P.$$

By the definition of $\tilde{\tau}_P^C$, $\tilde{\mathrm{can}}_P^C\circ f = P\otimes C$, hence $\tilde{\mathrm{can}}_P^C$ must be surjective and assertion follows by Theorem 2.2.7. □

Lemma 3.4.7. *If P is a left H-comodule, where H is a Hopf algebra, then for all $\sum_i p_i \otimes q_i \in P \otimes P$, the following two conditions are equivalent:*

$$\sum_i p_{i(-1)} q_{i(-1)} \otimes p_{i(0)} \otimes q_{i(0)} = 1_H \otimes \sum_i p_i \otimes q_i, \tag{3.42}$$

$$\sum_i p_{i(-1)} \otimes p_{i(0)} \otimes q_i = \sum_i S q_{i(-1)} \otimes p_i \otimes q_{i(0)}. \tag{3.43}$$

Proof. **(3.42)⇒(3.43)** Apply $(m \otimes P \otimes P) \circ (S \otimes {}^H\rho \otimes P)$ to both sides of (3.42).
(3.43)⇒(3.42)

$$\begin{aligned}
&\sum_i p_{i(-1)} q_{i(-1)} \otimes p_{i(0)} \otimes q_{i(0)} \\
&\qquad = [\text{ use (3.43) }] \sum_i S q_{i(-1)} q_{i(0)(-1)} \otimes p_i \otimes q_{i(0)(0)} \\
&\qquad\qquad = \sum_i S q_{i(-1)(1)} q_{i(-1)(2)} \otimes p_i \otimes q_{i(0)} = 1_H \otimes \sum_i p_i \otimes q_i.
\end{aligned}$$

□

The following two theorems state conditions for the existence and give an explicit form of the strong connection form.

Theorem 3.4.8. *Assume that:*

1. *$\mathbb{K}$ is a field, H is a Hopf algebra, and C is a coseparable coalgebra.*
2. *A is a right H-comodule algebra.*
3. *P is an (H, C)-bicomodule and a left H-comodule algebra. Also, there exists a grouplike $e \in C$ and a bijective entwining $\psi : C \otimes P \to P \otimes C$ such that $\rho^C(p) = \psi(e \otimes p)$, for any $p \in P$.*
4. *There exist coliftings $\tilde{\tau}_A^H : H \to A \otimes A$, $\tilde{\tau}_P^C : C \to P \otimes P$ of the translation maps which satisfy conditions (3.33-3.35) in Lemma 3.4.2.*

Then $(A \Box_H P)(R)^C$, where $R = (A \Box_H P)^{coC}$, is a principal extension.

Proof. Since $\mathbb{K}$ is a field, any $\mathbb{K}$-module is flat. Hence all the assumptions of Lemma 3.4.2 are satisfied and we know that there exists a colifting $\tilde{\tau}^C_{A \Box_H P} : C \to (A \Box_H P) \otimes (A \Box_H P)$ of the translation map. Moreover the existence of $\tilde{\tau}_P^C$ which satisfies condition (3.35) implies, by Lemma 3.4.4 and Lemma 3.4.7, that ψ commutes with the left H-coaction. Hence, by Lemma 3.4.5, there exists an invertible entwining $\psi_\Box$ on $A \Box_H P$ and $(A \Box_H P)_e(R, C, \psi_\Box)$ is an e-copointed $(A \Box_H P, C)_{\psi_\Box}$-extension. Then the assertion follows by Corollary 3.4.6. □

Theorem 3.4.9. *If*

1. *C is a coalgebra and H is a Hopf algebra with a bijective antipode,*
2. *A is a right H-comodule algebra and P is a left H-comodule algebra,*

3. *$P_e(B,C,\psi)$ is an e-copointed $(P,C)_\psi$-extension with a bijective entwining map, which commutes with the left H-coaction,*

4. *C, $A\otimes P$, $A\Box_H P$ are flat $\mathbb{K}$-modules,*

5. *there exist strong connection forms $\ell_A : H \to A\otimes A$ and $\ell_P : C \to P\otimes P$,*

6. *for all $c\in C$,*

$$c^{\underline{[1]}}{}_{(-1)} \otimes c^{\underline{[1]}}{}_{(0)} \otimes c^{\underline{[2]}} = Sc^{\underline{[2]}}{}_{(-1)} \otimes c^{\underline{[1]}} \otimes c^{\underline{[2]}}{}_{(0)}, \tag{3.44}$$

where $c^{\underline{[1]}} \otimes c^{\underline{[2]}} = \ell_P(c)$,

then $(A\Box_H P)_e(R,C,\psi_\Box)$, where $R = (A\Box_H P)^{coH}$, is an e-copointed $(A\Box_H P, C)_{\psi_\Box}$-extension with a bijective entwining (Lemma 3.4.5), and

$$\begin{aligned} &\ell_{A\Box_H P} : C \to (A\Box_H P)\otimes(A\Box_H P),\\ &c \mapsto (c^{\underline{[2]}}{}_{(-1)}{}^{\underline{[1]}} \otimes c^{\underline{[1]}}) \otimes (c^{\underline{[2]}}{}_{(-1)}{}^{\underline{[2]}} \otimes c^{\underline{[2]}}{}_{(0)}) \end{aligned} \tag{3.45}$$

is a strong connection form.

Note that while Definition 2.2.3 assumes that strong connection forms are defined for coalgebra Galois extensions, the actual definition of a strong connection form as a map satisfying conditions (2.17)-(2.20) does not require the Galois condition.

Proof. By Lemma 3.4.3, the left H-coaction commutes with the right C-coaction. Observe that had we assumed that P is an (H,C)-bicomodule, then commuting of ψ with the H-coaction would follow from (3.44) and Lemmas 3.4.7 and 3.4.4. Hence, by Lemma 3.4.5, $(A\Box_H P)_e(R,C,\psi_\Box)$ is an e-copointed $(A\Box_H P,C)_{\psi_\Box}$-extension with a bijective entwining.

In particular, ℓ_A and ℓ_P are coliftings of translation map, which satisfy all of the assumptions of Lemma 3.4.2, hence, by Lemma 3.4.2, the map $\ell_{A\Box_H P}$ given by (3.45) is a well defined colifting of the translation map on $A\Box_H P$. It remains to prove that (3.45) satisfies the remaining axioms (2.17, 2.19, 2.20) of a strong connection form. First compute

$$\begin{aligned} \ell_{A\Box_H P}(e) &= (e^{\underline{[2]}}{}_{(-1)}{}^{\underline{[1]}} \otimes e^{\underline{[1]}}) \otimes (e^{\underline{[2]}}{}_{(-1)}{}^{\underline{[2]}} \otimes e^{\underline{[2]}}{}_{(0)})\\ &= (1_{P(-1)}{}^{\underline{[1]}} \otimes 1_P) \otimes (1_{P(-1)}{}^{\underline{[2]}} \otimes 1_{P(0)}) = (1_H{}^{\underline{[1]}} \otimes 1_P)\otimes(1_H{}^{\underline{[2]}}\otimes 1_P)\\ &= (1_A\otimes 1_P)\otimes(1_A\otimes 1_P) = 1_{A\Box_H P}\otimes 1_{A\Box_H P}, \end{aligned}$$

where we used that $\ell_A(1_A) = 1_A\otimes 1_A$ and $\ell_P(e) = 1_P\otimes 1_P$. Hence, the map $\ell_{A\Box_H P}$ is normalised on e as required for (2.17).

Take any $c\in C$ and compute,

$$\begin{aligned} ((A\otimes P)\otimes\rho^C)\circ\ell_{A\Box_H P}(c) &= (c^{\underline{[2]}}{}_{(-1)}{}^{\underline{[1]}} \otimes c^{\underline{[1]}}) \otimes (c^{\underline{[2]}}{}_{(-1)}{}^{\underline{[2]}} \otimes c^{\underline{[2]}}{}_{(0)}) \otimes c^{\underline{[2]}}{}_{(1)}\\ &= (c^{\underline{[2]}}{}_{(0)(-1)}{}^{\underline{[1]}} \otimes c^{\underline{[1]}}) \otimes (c^{\underline{[2]}}{}_{(0)(-1)}{}^{\underline{[2]}} \otimes c^{\underline{[2]}}{}_{(0)(0)}) \otimes c^{\underline{[2]}}{}_{(1)}\\ \text{[by (2.20) for } \ell_P] &= (c_{(1)}{}^{\underline{[2]}}{}_{(-1)}{}^{\underline{[1]}} \otimes c_{(1)}{}^{\underline{[1]}}) \otimes (c_{(1)}{}^{\underline{[2]}}{}_{(-1)}{}^{\underline{[2]}} \otimes c_{(1)}{}^{\underline{[2]}}{}_{(0)}) \otimes c_{(2)}\\ &= \ell_{A\Box_H P}(c_{(1)})\otimes c_{(2)}. \end{aligned}$$

Therefore $\ell_{A\Box_H P}$ is a right C-comodule map, i.e., the condition (2.20) is satisfied. Finally, for all $c \in C$,

$$\begin{aligned}({}^C\psi\Box\rho \otimes (A \otimes P)) \circ \ell_{A\Box_H P}(c) &= e_A \otimes (c^{\underline{[2]}}{}_{(-1)}{}^{\underline{[1]}} \otimes c^{\underline{[1]}A}) \otimes (c^{\underline{[2]}}{}_{(-1)}{}^{\underline{[2]}} \otimes c^{\underline{[2]}}{}_{(0)}) \\ \text{[use (2.19) for } \ell_P] &= c_{(1)} \otimes (c_{(2)}{}^{\underline{[2]}}{}_{(-1)}{}^{\underline{[1]}} \otimes c_{(2)}{}^{\underline{[1]}}) \otimes (c_{(2)}{}^{\underline{[2]}}{}_{(-1)}{}^{\underline{[2]}} \otimes c_{(2)}{}^{\underline{[2]}}{}_{(0)}) \\ &= c_{(1)} \otimes \ell_{A\Box_H P}(c_{(2)}).\end{aligned}$$

Therefore, $\ell_{A\Box_H P}$ is a left C-comodule map, i.e., the condition (2.19) is satisfied. Thus we conclude that $\ell_{A\Box_H P}$ is a strong connection form as required. □

Let $P(B)^C_e$ be a C-coalgebra Galois extension, with a strong connection form $\ell : C \to P \otimes P$, $c \mapsto c^{\underline{[1]}} \otimes c^{\underline{[2]}}$. Clearly, the map

$$(p \otimes c) \mapsto pc^{\underline{[1]}} \otimes_B c^{\underline{[2]}} : P \otimes C \to P \otimes_B P$$

is a right inverse of can^C_P. But can^C_P is invertible, hence the right inverse of can^C_P must be equal to $(\mathrm{can}^C_P)^{-1}$. Hence, an explicit formula for the inverse of the canonical map is determined by the explicit formula for a strong connection form and by the knowledge of the subalgebra of coinvariants. The next lemma determines the subalgebra of coinvariants of $A\Box_H P$.

Lemma 3.4.10. *Suppose that H is a bialgebra and C is a coalgebra, both flat as a $\mathbb{K}$-modules. Let A be a right H-comodule algebra and let P be a left H-comodule algebra and an (H,C)-bicomodule. Moreover, suppose that $(P,C)_\psi$ is an entwining structure such that P is an entwined module and ψ commutes with the left H-coaction. Then*

$$(A\Box_H P)^{coC} = A\Box_H(P^{coC}). \tag{3.46}$$

Proof. As P is an entwined module, subalgebra of coinvariants is uniquely determined by the exactness of the following sequence:

$$0 \longrightarrow P^{\mathrm{co}C} \longrightarrow P \underset{p\mapsto p1_{(0)}\otimes 1_{(1)}}{\overset{\rho^C}{\rightrightarrows}} P \otimes C. \tag{3.47}$$

View $P \otimes C$ as a left H-comodule by the left H-coaction on the first factor. The functor $A\Box_H\cdot : {}^H\mathcal{M} \to \mathcal{M}$ is left exact, and, furthermore, since C is flat as a $\mathbb{K}$-module, $A\Box_H(P \otimes C) \simeq (A\Box_H P) \otimes C$ (cf. [41]). Hence, cotensoring the sequence (3.47) with A, yields the following exact sequence

$$0 \longrightarrow A\Box_H(P^{\mathrm{co}C}) \longrightarrow A\Box_H P \underset{A\otimes(p\mapsto p1_{(0)}\otimes 1_{(1)})}{\overset{A\otimes\rho^C}{\rightrightarrows}} (A\Box_H P) \otimes C. \tag{3.48}$$

By Lemmas 3.2.1 and 3.4.5, $A\Box_H P$ is an algebra and an entwined module, hence the sequence (3.48) defines uniquely the algebra of coinvariants $(A\Box_H P)^{\mathrm{co}C} = A\Box_H(P^{\mathrm{co}C})$. □

3.5 Example: The cotensor product of Matsumoto spheres II

Let us keep the notation from Section 3.3 unless indicated otherwise. There exists an alternative $\mathbb{Z}$-grading on P:

$$\bar{\deg}(a') = -1,\ \bar{\deg}(a'^*) = 1,\ \bar{\deg}(b') = 1,\ \bar{\deg}(b'^*) = -1. \tag{3.49}$$

For $n \in \mathbb{Z}$, let ${}_nP$ denote the degree n part of P with respect to the above grading. Let $\bar{H} = H$. We define an alternative left $\bar{H}$-coaction ${}^{\bar{H}}\rho : P \to H \otimes P$:

$${}^{\bar{H}}\rho(p) = u^{\bar{\deg}(p)} \otimes p \text{ for all homogeneous } p \in P, \tag{3.50}$$

where we use the bar over H to indicate it is coaction different from the one defined by (eq. 3.19).

Note that clearly

$$A\Box_{\bar{H}}P = \bigoplus_{n\in\mathbb{Z}} A_n \otimes {}_nP. \tag{3.51}$$

Let $\phi \in \mathbb{R}$. There exists an isomorphism of the $*$-algebras $f_\phi : \vartheta(S^3_\phi) \to \vartheta(S^3_{-\phi})$ defined on the generators by

$$f_\phi(b) = b^*,\ f_\phi(a) = a, \tag{3.52}$$

where we used the same symbols for the generators of $\vartheta(S^3_\phi)$ and $\vartheta(S^3_{-\phi})$. It follows that $A \otimes f_{-\theta'}$ is a $*$-algebra isomorphism from $A\Box_H\vartheta(S^3_{-\theta'})$ to $A\Box_{\bar{H}}P$. It is therefore clear that, as a $*$-algebra, $A\Box_{\bar{H}}P$ is generated by the elements

$$\alpha = a \otimes a^*, \qquad \beta = b \otimes b, \qquad \gamma = a \otimes b, \qquad \delta = b \otimes a^*, \tag{3.53}$$

which satisfy the following commutation relations

$$\begin{array}{llll}
\alpha\alpha^* = \alpha^*\alpha, & \beta\beta^* = \beta^*\beta, & \gamma\gamma^* = \gamma^*\gamma, & \delta\delta^* = \delta^*\delta, \\
\alpha\beta = \lambda\bar{\lambda}'\beta\alpha, & \alpha\beta^* = \bar{\lambda}\lambda'\beta^*\alpha, & \alpha\gamma = \bar{\lambda}'\gamma\alpha, & \alpha\gamma^* = \lambda'\gamma^*\alpha, \\
\alpha\delta = \lambda\delta\alpha, & \alpha\delta^* = \bar{\lambda}\delta^*\alpha, & \beta\gamma = \bar{\lambda}\gamma\beta, & \beta\gamma^* = \lambda\gamma^*\beta, \\
\beta\delta = \lambda'\delta\beta, & \beta\delta^* = \bar{\lambda}'\delta^*\beta, & \gamma\delta = \lambda\lambda'\delta\gamma, & \gamma\delta^* = \bar{\lambda}\bar{\lambda}'\delta^*\gamma,
\end{array} \tag{3.54}$$

and, in addition,

$$\alpha^*\alpha + \beta^*\beta + \gamma^*\gamma + \delta^*\delta = 1, \tag{3.55}$$

$$\alpha\beta = \bar{\lambda}'\gamma\delta, \tag{3.56}$$

where $\lambda = e^{2\pi i\theta}$ and $\lambda' = e^{2\pi i\theta'}$, and we used the same letters α, β, γ, δ as for the generators of $A\Box_{\bar{H}}P$ in Section 3.3.

By Lemma 3.4.10, the algebra of coinvariants $R = (A\Box_{\bar{H}}P)^{\mathrm{co}H} = A\Box_{\bar{H}}D$. For all $n \in 2\mathbb{Z}$, let ${}_nD$ denote the degree n part of D with respect to the restriction to D of the grading (3.49). Note that $\bar{\deg}(x'_\pm) = \pm 2$, $\bar{\deg}(z') = 0$. It is clear that

$$A\Box_H D = \bigoplus_{n\in 2\mathbb{Z}} A_n \otimes {}_nD.$$

Note that the monomials $(a^*a)^m a^\zeta b^\eta$, $m \in \mathbb{N}_0$, $\zeta, \eta \in \mathbb{Z}$, form a basis of A and the monomials $z'^n (x'_+)^\mu$, $n \in \mathbb{N}_0$, $\mu \in \mathbb{Z}$, form a basis of D. Therefore the family of monomials

$$(a^*a)^m a^\zeta b^\eta \otimes z'^n (x'_+)^\mu, \quad m, n \in \mathbb{N}_0,\ \zeta, \eta \in \mathbb{Z},\ \zeta + \eta = 2\mu \tag{3.57}$$

is a basis of $R = A \square_{\tilde{H}} D$ as a $\mathbb{C}$-vector space. Denote by

$$\begin{aligned} z^1 &= z \otimes 1 = \alpha^*\alpha + \gamma^*\gamma, \quad z^2 = 1 \otimes z' = \alpha\alpha^* + \delta\delta^*, \\ x^1_+ &= x_+ \otimes 1 = \delta\alpha^* + \beta\gamma^*, \quad x^1_- = x_- \otimes 1 = \alpha\delta^* + \gamma\beta^*, \\ x^a_+ &= aa \otimes x'_+ = \gamma\alpha, \quad x^a_- = a^*a^* \otimes x'_- = \alpha^*\gamma^*, \\ x^b_+ &= bb \otimes x'_+ = \beta\delta, \quad x^b_- = b^*b^* \otimes x'_- = \delta^*\beta^*. \end{aligned} \tag{3.58}$$

Observe that

$$ab = ab(aa^* + bb^*) = \lambda(ba^*)(aa) + (ab^*)(bb). \tag{3.59}$$

Suppose that $m, n \in \mathbb{N}_0$, $\zeta, \eta, \mu \in \mathbb{Z}$ and $\zeta + \eta = 2\mu$. It can easily be checked that the following equations are satisfied.

1. If $\zeta\eta > 0$ and ζ and η are even, then

$$(a^*a)^m a^\zeta b^\eta \otimes z'^n (x'_+)^\mu = (z^1)^m (z^2)^n (x^a_+)^{\frac{\zeta}{2}} (x^b_+)^{\frac{\eta}{2}}. \tag{3.60}$$

2. If $\zeta\eta > 0$ and ζ and η are odd, then, by (3.59),

$$\begin{aligned} (a^*a)^m a^\zeta b^\eta \otimes z'^n (x'_+)^\mu = (z^1)^m (z^2)^n \big(&\lambda^\zeta x^1_+ (x^a_+)^{\frac{\zeta+1}{2}} (x^b_+)^{\frac{\eta-1}{2}} \\ &+ \lambda^{-\zeta+1} x^1_- (x^a_+)^{\frac{\zeta-1}{2}} (x^b_+)^{\frac{\eta+1}{2}}\big). \end{aligned} \tag{3.61}$$

3. If $\zeta\eta \leq 0$ and $|\zeta| \leq |\eta|$, then, by (3.22),

$$(a^*a)^m a^\zeta b^\eta \otimes z'^n (x'_+)^\mu = \lambda^{-\frac{\zeta(\zeta-1)}{2}} (z^1)^m (z^2)^n (x^1_-)^\zeta (x^b_+)^\mu. \tag{3.62}$$

4. If $\zeta\eta \leq 0$ and $|\zeta| > |\eta|$, then, by (3.22),

$$(a^*a)^m a^\zeta b^\eta \otimes z'^n (x'_+)^\mu = \lambda^{\frac{\eta(\eta-1)}{2} + \zeta\eta} (z^1)^m (z^2)^n (x^1_+)^\eta (x^a_+)^\mu. \tag{3.63}$$

Hence it is clear that the elements z^1, z^2, $x^1_\pm$, $x^a_\pm$, $x^b_\pm$ generate $R = A \square_{\tilde{H}} D$ as a *-algebra. One also easily checks that they satisfy the following commutation relations:

$$\begin{aligned} x^1_+ x^1_- &= x^1_- x^1_+, & x^a_+ x^a_- &= x^a_- x^a_+, & x^b_+ x^b_- &= x^b_- x^b_+, \\ x^1_+ x^a_+ &= \bar{\lambda}^2 x^a_+ x^1_+, & x^1_+ x^a_- &= \lambda^2 x^a_- x^1_+, & x^1_+ x^b_+ &= \bar{\lambda}^2 x^b_+ x^1_+, \\ x^1_+ x^b_- &= \lambda^2 x^b_- x^1_+, & x^a_+ x^b_+ &= \lambda^4 x^b_+ x^a_+, & x^a_+ x^b_- &= \bar{\lambda}^4 x^b_- x^a_+, \end{aligned} \tag{3.64}$$

and z^1 and z^2 are central in R. In addition to relations (3.64), the generators z^1, z^2, $x^1_\pm$, $x^a_\pm$, $x^b_\pm$, satisfy the following 'geometric' relations:

$$x^1_+x^1_- + (z^1)^2 = z^1, \tag{3.65}$$
$$x^a_+x^a_- = (z^1)^2z^2(1-z^2), \tag{3.66}$$
$$x^b_+x^b_- = (1-z^1)^2z^2(1-z^2), \tag{3.67}$$
$$x^a_+x^b_- = \bar{\lambda}(x^1_-)^2z^2(1-z^2), \tag{3.68}$$

and the following 'auxiliary' relations:

$$\lambda(1-z^1)x^1_+x^a_+ = z^1x^1_-x^b_+, \tag{3.69}$$
$$\lambda(1-z^1)^2x^a_+ = (x^1_-)^2x^b_+, \tag{3.70}$$
$$\lambda(x^1_+)^2x^a_+ = (z^1)^2x^b_+. \tag{3.71}$$

Denote by $\bar{R}$ the quotient of a free algebra generated by the elements z^1, z^2, $x^1_\pm$, $x^a_\pm$, $x^b_\pm$ by the ideal generated by the relations (3.64)-(3.71). By definition, there exists an algebra surjection $\bar{R} \to R$, mapping the generators of $\bar{R}$ to the respective generators of R. Therefore, in order to prove that R and $\bar{R}$ are mutually isomorphic, it is enough to prove that, as a $\mathbb{C}$-vector space, $\bar{R}$ is spanned by the expressions occurring on the right hand sides of the equations (3.60)-(3.63).

It is immediately clear that the monomials

$$(z^1)^m(z^2)^n(x^1_+)^\mu(x^a_+)^\varsigma(x^b_+)^\upsilon,\ m,n \in \mathbb{N}_0,\ \mu,\varsigma,\upsilon \in \mathbb{Z},\ \varsigma\upsilon \geq 0, \tag{3.72}$$

span $\bar{R}$. In R, we can express monomials (3.72) in terms of the right hand sides of the equations (3.60-3.63) by substituting (3.58) and then using equations (3.60-3.63). One can check that thus obtained formulae follow from the relations (3.64)-(3.71). Therefore, we also expressed monomials (3.72), viewed as elements of $\bar{R}$, in terms of the right hand sides of the equations (3.60)-(3.63), viewed as elements of $\bar{R}$, which ends the proof.

The above procedure requires dividing the set of possible combinations of values of indices μ, ς, υ in (3.72) into groups which allow for using one of the formulae (3.60-3.63). The following is the basic division.

Case 1: $\mu = 0$. **Case 2:** $\mu \neq 0, \varsigma = 0, \mu\upsilon \geq 0$.
Case 3: $\mu \neq 0, \mu\varsigma \leq 0, \mu\upsilon < 0$. **Case 4:** $\mu \neq 0, \mu\varsigma > 0, \mu\upsilon \geq 0$.
Case 5: $\mu \neq 0, \mu\varsigma \leq 0, \upsilon = 0$.

Note that 'Case 1' (resp. 'Case 2', 'Case 5') monomials (3.72) are already in the form of the right hand side of the equation (3.60) (resp. (3.62), (3.63)). Cases 3 and 4 branch into further subcases, but those subdivisions are obvious as soon as one writes the formulae, so we do not write them explicitly here. As an example, we will prove that one of the subsets of 'Case 4' monomials (3.72) is spanned by the right hand sides of ((3.60)-(3.63)). We leave the other cases to the reader.

Substituting (3.58) to (3.72) yields

$$(z^1)^m(z^2)^n(x^1_+)^\mu(x^a_+)^\varsigma(x^b_+)^\upsilon = \lambda^{\frac{\mu(\mu+1)}{2}-2\varsigma\mu}(a^*a)^m a^{-\mu}a^{2\varsigma}b^\mu b^{2\upsilon} \otimes z'^n(x'_+)^{\varsigma+\upsilon} = (*).$$

Suppose that indices $\mu, \varsigma, \upsilon \in \mathbb{Z}$ belong to Case 4. There are two subcases: $|\mu| \geq |2\varsigma|$ and $|\mu| < |2\varsigma|$. We shall examine the second possibility. In this case

$$(*) = \lambda^{\frac{\mu(\mu+1)}{2} - 2\varsigma\mu}(a^*a)^{m+|\mu|}a^{2\varsigma-\mu}b^{2\upsilon+\mu} \otimes z'^n(x'_+)^{\varsigma+\upsilon} = (**).$$

Note that it is enough to consider only the case $\mu > 0$, as the case with $\mu < 0$ is the *-conjugate of the former. There are again two cases. If μ is even, we can apply (3.60). If μ is odd, we can apply (3.61). In the latter case

$$(**) = \lambda^{\frac{\mu(\mu+1)}{2} - 2\varsigma\mu}(z^1)^{m+\mu}(z^2)^n(\lambda^{2\varsigma-\mu}x^1_+(x^a_+)^{\varsigma-\frac{\mu-1}{2}}(x^b_+)^{\upsilon+\frac{\mu-1}{2}} \\ + \lambda^{\mu-2\varsigma+1}x^1_-(x^a_+)^{\varsigma-\frac{\mu+1}{2}}(x^b_+)^{\upsilon+\frac{\mu+1}{2}}).$$

We are to prove that the equation in R, obtained above, is also satisfied in $\bar{R}$. By factoring out the common factors $(x^a_+)^{\varsigma-\frac{\mu+1}{2}}(x^b_+)^{\upsilon}$ (we assumed above that $\mu > 0$) on both sides of the above formula, one sees that it is equivalent to the following equation in $\bar{R}$:

$$(x^1_+)^\mu(x^a_+)^{\frac{\mu+1}{2}} = \lambda^{-\frac{\mu(\mu+1)}{2}}(z^1)^\mu(\lambda x^1_+x^a_+ + x^1_-x^b_+)(x^b_+)^{\frac{\mu-1}{2}}. \tag{3.73}$$

Note that $(z^1)^\mu(x^b_+)^{\frac{\mu-1}{2}} = z^1((z^1)^2x^b_+)^{\frac{\mu-1}{2}}$. Hence, by (3.71), equation (3.73) is equivalent to

$$(z^1x^1_-x^b_+ - \lambda(1-z^1)x^1_+x^a_+)(x^1_+)^{\mu-1}(x^a_+)^{\frac{\mu-1}{2}} = 0.$$

But, by (3.69) the expression in parenthesis is equal to zero. Hence the above equation is satisfied in $\bar{R}$.

To gain a better geometric understanding of the algebra $A\Box_H D$, we look at the classical case, whereby $\theta = \theta' = 0$. In accordance with the remarks in the introduction about the classical interpretation of the cotensor product, the algebra of the coinvariants $A\Box_H D$ is the commutative algebra of functions on a fibered space with the base S^2 and the fibre S^2. Denote this space by $S^2_{S^2}$, i.e.,

$$S^2_{S^2} = \{(x^1_+, x^a_+, x^b_+, z^1, z^2) \in \mathbb{C}^3 \times \mathbb{R}^2 |\ \text{eq's (3.65)-(3.68) are satisfied}\},$$

where we set $\lambda = \lambda' = 1$ in equations (3.65)-(3.68).

Observe that the 'auxiliary' equations (3.69)-(3.71) with θ set to zero, follow from equations (3.65)-(3.68). Indeed, multiplying (3.67) by $x^1_+x^a_+$, and then using (3.65) and (3.68) yields

$$z^2(1-z^2)(1-z^1)^2x^1_+x^a_+ = z^2(1-z^2)(1-z^1)z^1x^1_-x^b_+,$$

which is a multiple of (3.69) by $z^2(1-z^2)(1-z^1)$. Note that, by (3.66) and (3.67), if $z^2(1-z^2) = 0$ then $x^a_+ = 0$ and $x^b_+ = 0$, and then equation (3.69) is satisfied. Similarly, if $z^1 = 1$, then $x^b_+ = 0$ and the equation (3.69) is also satisfied. Similarly we derive the formulae (3.70) and (3.71).

It is easy to see that the parameters $x^1_\pm$, z^1 and the relation (3.65) describe the base space of the fibre bundle $S^2_{S^2}$. Hence we can define a surjection on the base space $\pi : S^2_{S^2} \to S^2$, by

$$\pi : (x^1_+, x^a_+, x^b_+, z^1, z^2) \mapsto (x^1_+, z^1).$$

Define also $U_0, U_1 \subset S^2$, $U_k = \{(x_+, z) \in S^2 | z \neq k\}$.

For any parameter $\alpha \in \mathbb{R}_+$, the equation

$$x_+ x_- = \alpha z(1-z),$$

where $z \in \mathbb{R}$, $x_- = \overline{x_+} \in \mathbb{C}$, describes an ellipsoid with the equatorial radius $\frac{\sqrt{\alpha}}{2}$, and the longitudal diameter 1, which is obviously homeomorphic to S^2 by the substitution $(x_+, z) \mapsto (\frac{x_+}{\sqrt{\alpha}}, z)$. Define two continuous maps $\phi_0 : \pi^{-1}(U_0) \to U_0 \times S^2$ and $\phi_1 : \pi^{-1}(U_1) \to U_0 \times S^2$, by

$$\phi_0 : (x^1_+, x^a_+, x^b_+, z^1, z^2) \mapsto ((x^1_+, z^1), (\frac{x^a_+}{z^1}, z^2)),$$
$$\phi_1 : (x^1_+, x^a_+, x^b_+, z^1, z^2) \mapsto ((x^1_+, z^1), (\frac{x^b_+}{1-z^1}, z^2)).$$

These maps have the inverses

$$(\phi_0)^{-1} : ((x^1_+, z^1), (x_+, z)) \mapsto (x^1_+, x^a_+, x^b_+, z^1, z^2) = (x^1_+, z^1 x_+, (x^1_+)^2 \frac{x_+}{z^1}, z^1, z),$$

and

$$(\phi_1)^{-1} : ((x^1_+, z^1), (x_+, z)) \mapsto (x^1_+, x^a_+, x^b_+, z^1, z^2)$$
$$= (x^1_+, (x^1_-)^2 \frac{x_+}{1-z^1}, (1-z^1)x_+, z^1, z),$$

which are also continuous. Hence, the maps ϕ_0 and ϕ_1 are homeomorphisms which define the local trivialisations of the fibre bundle $S^2_{S^2}$. Moreover they cannot be extended to the whole of $S^2_{S^2}$, hence it is our conjecture that $S^2_{S^2}$ is a nontrivial fibre bundle, though we were unable to prove it strictly. $A \Box_H D$ is a quantum deformation of the algebra of functions on the fibre bundle $S^2_{S^2}$.

For any $\theta \in \mathbb{R}$, the H-Hopf Galois extension $\vartheta(S^3_\theta)(\vartheta(S^2))^H$ admits a strong connection form given explicitly by

$$\ell(u^n) = \sum_{m=0}^{n} \binom{n}{m} b^{*m} a^{*n-m} \otimes a^{n-m} b^m, \tag{3.74}$$

$$\ell(u^{*n}) = \sum_{m=0}^{n} \binom{n}{m} b^m a^{n-m} \otimes a^{*n-m} b^{*m}, \tag{3.75}$$

for all $n \in \mathbb{N}$ (cf. [12]). Hence $A = \vartheta(S^3_\theta)$ and $P = \vartheta(S^3_{\theta'})$ admit strong connection forms ℓ_A and ℓ_P, given by the above formulae. Observe that ℓ_P satisfies condition

(3.44) of Theorem 3.4.9. Hence, by Theorem 3.4.9, there exists a strong connection form $\ell_{A\Box_{\tilde{H}}P}$ on $A\Box_{\tilde{H}}P$. Explicitly, by (3.45), (3.74) and (3.75), for any $n \in \mathbb{N}$,

$$\begin{aligned}\ell_{A\Box_H P}(u^n) = & \sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{k=0}^{n-2m}\binom{n}{m}\binom{n-2m}{k}(b^k a^{n-2m-k}\otimes b'^{*m}a'^{*n-m}) \\ & \otimes (a^{*n-2m-k}b^{*k}\otimes a'^{n-m}b'^{m}) \\ & + \sum_{m=\lfloor\frac{n}{2}\rfloor+1}^{n}\sum_{k=0}^{2m-n}\binom{n}{m}\binom{2m-n}{k}(b^{*k}a^{*2m-n-k}\otimes b'^{*m}a'^{*n-m}) \\ & \otimes (a^{2m-n-k}b^{k}\otimes a'^{n-m}b'^{m}),\end{aligned} \tag{3.76}$$

and

$$\begin{aligned}\ell_{A\Box_H P}(u^{-n}) = & \sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{k=0}^{n-2m}\binom{n}{m}\binom{n-2m}{k}(b^{*k}a^{*n-2m-k}\otimes b'^{m}a'^{n-m}) \\ & \otimes (a^{n-2m-k}b^{k}\otimes a'^{*n-m}b'^{*m}) \\ & + \sum_{m=\lfloor\frac{n}{2}\rfloor+1}^{n}\sum_{k=0}^{2m-n}\binom{n}{m}\binom{2m-n}{k}(b^{k}a^{2m-n-k}\otimes b'^{m}a'^{n-m}) \\ & \otimes (a^{*2m-n-k}b^{*k}\otimes a'^{*n-m}b'^{*m}).\end{aligned} \tag{3.77}$$

In order to express the above formulae in terms of the generators (3.53), observe that

$$\begin{aligned}a^{*n-2m-k}b^{*k}\otimes a'^{n-m}b'^{m} &= a^{*n-2m-k}b^{*k}(a^*a+b^*b)^m\otimes a'^{n-m}b'^{m} \\ &= \sum_{t=0}^{m}\binom{m}{t}\bar{\lambda}^{t(k+m-t)}a^{*n-2m-k+t}b^{*m+k-t}a^t b^{m-t}\otimes a'^{n-m}b'^{m} \\ &= \sum_{t=0}^{m}\binom{m}{t}\bar{\lambda}^{t(k+m-t)}\alpha^{*n-2m-k+t}\delta^{*m+k-t}\gamma^t\beta^{m-t}.\end{aligned}$$

Similarly, consider the second factor of the tensor product in the second summand in (3.76). If $0 \le k \le 2m-n$ and $\lfloor\frac{n}{2}\rfloor < m \le n$, then $k \le m$. It follows that

$$\begin{aligned}a^{2m-n-k}b^{k}\otimes a'^{n-m}b'^{m} &= (a^*a+b^*b)^{n-m}a^{2m-n-k}b^{k}\otimes a'^{n-m}b'^{m} \\ &= \sum_{t=0}^{n-m}\binom{n-m}{t}a^{*t}b^{*n-m-t}b^{n-m-t}a^{2m-n-k+t}b^{k}\otimes a'^{n-m}b'^{m} \\ &= \sum_{t=0}^{n-m}\binom{n-m}{t}\lambda^{(2m-n-k+t)k}\alpha^{*t}\delta^{*n-m-t}\beta^{n-m-t+k}\gamma^{2m-n-k+t}.\end{aligned}$$

Hence

$$
\begin{aligned}
\ell_{A\Box_H P}(u^n) = &\sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{k=0}^{n-2m} \sum_{t=0}^{m} \sum_{s=0}^{m} \binom{n}{m} \binom{n-2m}{k} \binom{m}{t} \binom{m}{s} \lambda^{(k+m)(t-s)-t^2+s^2} \\
&\cdot \beta^{*m-t} \gamma^{*t} \delta^{k+m-t} \alpha^{n-2m-k+t} \otimes \alpha^{*n-2m-k+s} \delta^{*k+m-s} \gamma^{s} \beta^{m-s} \\
&+ \sum_{m=\lfloor \frac{n}{2} \rfloor+1}^{n} \sum_{k=0}^{2m-n} \sum_{t=0}^{n-m} \sum_{s=0}^{n-m} \binom{n}{m} \binom{2m-n}{k} \binom{n-m}{t} \binom{n-m}{s} \bar{\lambda}^{k(t-s)} \\
&\cdot \gamma^{*2m-n-k+t} \beta^{*n-m+k-t} \delta^{n-m-t} \alpha^{t} \\
&\otimes \alpha^{*s} \delta^{*n-m-s} \beta^{n-m-s+k} \gamma^{2m-n-k+s}.
\end{aligned}
\tag{3.78}
$$

Similarly we prove that

$$
\begin{aligned}
\ell_{A\Box_H P}(u^{-n}) = &\sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{k=0}^{n-2m} \sum_{t=0}^{m} \sum_{s=0}^{m} \binom{n}{m} \binom{n-2m}{k} \binom{m}{t} \binom{m}{s} \lambda^{(k+m)(t-s)-t^2+s^2} \\
&\cdot \alpha^{*n-2m-k+s} \delta^{*k+m-s} \gamma^{s} \beta^{m-s} \otimes \beta^{*m-t} \gamma^{*t} \delta^{k+m-t} \alpha^{n-2m-k+t} \\
&+ \sum_{m=\lfloor \frac{n}{2} \rfloor+1}^{n} \sum_{k=0}^{2m-n} \sum_{t=0}^{n-m} \sum_{s=0}^{n-m} \binom{n}{m} \binom{2m-n}{k} \binom{n-m}{t} \binom{n-m}{s} \bar{\lambda}^{k(t-s)} \\
&\cdot \alpha^{*s} \delta^{*n-m-s} \beta^{n-m-s+k} \gamma^{2m-n-k+s} \\
&\otimes \gamma^{*2m-n-k+t} \beta^{*n-m+k-t} \delta^{n-m-t} \alpha^{t}.
\end{aligned}
\tag{3.79}
$$

Since $H = \vartheta(U(1))$ is a coseparable Hopf algebra, Theorem 3.4.8 implies that $(A\Box_{\bar{H}}P)(A\Box_{\bar{H}}D)^H$ is an H-Hopf Galois extension.

3.6 Cotensor products and cleft extensions

In the present section we examine the case of cotensor products of cleft extensions.

Proposition 3.6.1. *Assume that*

1. *H is a Hopf algebra and C is a coalgebra flat as a $\mathbb{K}$-module.*

2. *A is a right H-comodule algebra and P is an (H,C)-bicomodule and a left H-comodule algebra.*

3. *There exist cleaving maps $\gamma_A : H \to A$, $\gamma_P : C \to P$.*

4. *There exists a coalgebra map $f : C \to H$ such that, for any $c \in C$,*

$$
{}^{H}\rho(\gamma_P(c)) = f(c_{(1)}) \otimes \gamma_P(c_{(2)}). \tag{3.80}
$$

Then linear functions $\gamma_{A\Box_H P}, \gamma^{-1}_{A\Box_H P} : C \to A\Box_H P$, *given explicitly as*

$$\gamma_{A\Box_H P}(c) = \gamma_A(f(c_{(1)})) \otimes \gamma_P(c_{(2)}), \tag{3.81}$$
$$\gamma^{-1}_{A\Box_H P}(c) = \gamma_A^{-1}(f(c_{(2)})) \otimes \gamma_P^{-1}(c_{(1)}), \tag{3.82}$$

for all $c \in C$, *are, respectively, the cleaving map and its inverse. Furthermore, if* P *satisfies any of the equivalent conditions in Proposition 2.5.3 (for instance assume that* $P(D)^C_{\gamma_P}$ *is a* C*-coalgebra Galois extension, where* $D = P^{coC}$*), then* $(A\Box_H P)(R)^C_{\gamma_{A\Box_H P}}$*, where* $R = (A\Box_H P)^{coC}$*, is a* C*-coalgebra Galois extension.*

Proof. We first prove that $\gamma_{A\Box_H P}(C), \gamma^{-1}_{A\Box_H P}(C) \subseteq A\Box_H P$. Indeed, for any $c \in C$,

$$\begin{aligned}(\rho^H \otimes P) \circ \gamma_{A\Box_H P}(c) &= \gamma_A(f(c_{(1)}))_{(0)} \otimes \gamma_A(f(c_{(1)}))_{(1)} \otimes \gamma_P(c_{(2)}) \\ &= \gamma_A(f(c_{(1)})) \otimes f(c_{(2)}) \otimes \gamma_P(c_{(3)}) = (A \otimes {}^H\rho) \circ \gamma_{A\Box_H P}(c),\end{aligned}$$

and

$$\begin{aligned}(\rho^H \otimes P) \circ \gamma^{-1}_{A\Box_H P}(c) &= \gamma_A^{-1}(f(c_{(2)}))_{(0)} \otimes \gamma_A^{-1}(f(c_{(2)}))_{(1)} \otimes \gamma_P^{-1}(c_{(1)}) \\ [\text{use (2.57) for } \gamma_A^{-1}] &= \gamma_A^{-1}(f(c_{(3)})) \otimes Sf(c_{(2)}) \otimes \gamma_P^{-1}(c_{(1)}) \\ [\text{use (2.61) for } \gamma_P^{-1}] &= (A \otimes {}^H\rho) \circ \gamma^{-1}_{A\Box_H P}(c).\end{aligned}$$

By Lemma 3.2.1, $A\Box_H P$ is an algebra and a right C-comodule. For any $c \in C$,

$$\begin{aligned}&\gamma_{A\Box_H P}(c_{(1)})\gamma^{-1}_{A\Box_H P}(c_{(2)}) = (\gamma_A(f(c_{(1)})) \otimes \gamma_P(c_{(2)}))(\gamma_A^{-1}(f(c_{(4)})) \otimes \gamma_P^{-1}(c_{(3)})) \\ &= \gamma_A(f(c_{(1)}))\gamma_A^{-1}(f(c_{(4)})) \otimes \gamma_P(c_{(2)})\gamma_P^{-1}(c_{(3)}) = \gamma_A(f(c_{(1)}))\gamma_A^{-1}(f(c_{(2)})) \otimes 1_P \\ &= \varepsilon(f(c))1_A \otimes 1_P = \varepsilon(c)1_{A\Box_H P},\end{aligned}$$

and

$$\begin{aligned}&\gamma^{-1}_{A\Box_H P}(c_{(1)})\gamma_{A\Box_H P}(c_{(2)}) = (\gamma_A^{-1}(f(c_{(2)})) \otimes \gamma_P^{-1}(c_{(1)}))(\gamma_A(f(c_{(3)})) \otimes \gamma_P(c_{(4)})) \\ &= \gamma_A^{-1}(f(c_{(2)}))\gamma_A(f(c_{(3)})) \otimes \gamma_P^{-1}(c_{(1)})\gamma_P(c_{(4)}) = 1_A \otimes \gamma_P^{-1}(c_{(1)})\gamma_P(c_{(2)}) \\ &= \varepsilon(c)1_A \otimes 1_P = \varepsilon(c)1_{A\Box_H P},\end{aligned}$$

hence $\gamma^{-1}_{A\Box_H P}$ is the convolution inverse of $\gamma_{A\Box_H P}$ as claimed. Moreover, $\gamma_{A\Box_H P}$ is also a right colinear map:

$$\begin{aligned}\rho^C \circ \gamma_{A\Box_H P}(c) &= \gamma_A(f(c_{(1)})) \otimes \gamma_P(c_{(2)})_{(0)} \otimes \gamma_P(c_{(2)})_{(1)} \\ &= \gamma_A(f(c_{(1)})) \otimes \gamma_P(c_{(2)}) \otimes c_{(3)} = \gamma_{A\Box_H P}(c_{(1)}) \otimes c_{(2)},\end{aligned}$$

for any $c \in C$.

Assume that P satisfies one of the equivalent conditions of Proposition 2.5.3. In particular, by Proposition 2.5.3, there exists an entwining $\psi : C \otimes P \to P \otimes C$ such that P is a $(P, C)_\psi$-entwined module. Observe that $\gamma_P^{-1}(c_{(1)}) \otimes \gamma_P(c_{(2)})$ is a colifting of the translation map on P, which, by (2.62), satisfies the assumptions of Lemma 3.4.4, and hence, by Lemma 3.4.4, the entwining ψ commutes with the left H-coaction. This implies that, by Lemma 3.4.5, there exists an entwining $\psi_\square$ such that $A\square_H P \in \mathcal{M}^C_{A\square_H P}(\psi_\square)$. Hence, by Proposition 2.5.3, $A\square_H P(R)^C_{\gamma_{A\square_H P}}$, where $R = (A\square_H P)^{coC}$, is a cleft C-coalgebra Galois extension. $\square$

Proposition 3.6.2. *Suppose that*

1. *H is a bialgebra and C is a coalgebra flat as a $\mathbb{K}$-module;*
2. *A is a right H-comodule algebra and P is an (H,C)-bicomodule and a left H-comodule algebra;*
3. *there exists a cleaving map $\gamma_P : C \to P$;*
4. *There exists a linear map $\gamma_A : H \to A$, which is unital (i.e. $\gamma_A(1_H) = 1_A$), multiplicative (i.e. $\gamma_A(hh') = \gamma_A(h)\gamma_A(h')$, for all $h, h' \in H$), and right H-colinear (i.e. $\rho^H(\gamma_A(h)) = \gamma_A(h_{(1)}) \otimes h_{(2)}$, for all $h \in H$).*

Then linear functions $\gamma_{A\square_H P}, \gamma^{-1}_{A\square_H P} : C \to A\square_H P$, given explicitly for any $c \in C$ as

$$\gamma_{A\square_H P}(c) = \gamma_A(\gamma_P(c)_{(-1)}) \otimes \gamma_P(c)_{(0)}, \tag{3.83}$$
$$\gamma^{-1}_{A\square_H P}(c) = \gamma_A(\gamma_P^{-1}(c)_{(-1)}) \otimes \gamma_P^{-1}(c)_{(0)}, \tag{3.84}$$

are a cleaving map and its inverse, respectively.

Moreover, if there exists an entwining map $\psi : C \otimes P \to P \otimes C$, such that P is a $(P,C)_\psi$-entwined module and the entwining map ψ commutes with the left H-coaction, then $(A\square_H P)(R)^C_{\gamma_{A\square_H P}}$, where $R = (A\square_H P)^{coC}$, is a C-coalgebra Galois extension.

Proof. We first prove that $\gamma_{A\square_H P}(C), \gamma^{-1}_{A\square_H P}(C) \subseteq A\square_H P$. Indeed, for all $c \in C$,

$$\begin{aligned}(\rho^H \otimes P) \circ \gamma_{A\square_H P}(c) &= \gamma_A(\gamma_P(c)_{(-1)})_{(0)} \otimes \gamma_A(\gamma_P(c)_{(-1)})_{(1)} \otimes \gamma_P(c)_{(0)} \\ &= \gamma_A(\gamma_P(c)_{(-2)}) \otimes \gamma_P(c)_{(-1)} \otimes \gamma_P(c)_{(0)} = (A \otimes {}^H\rho) \circ \gamma_{A\square_H P}(c)\end{aligned}$$

and

$$\begin{aligned}(\rho^H \otimes P) \circ \gamma^{-1}_{A\square_H P}(c) &= \gamma_A(\gamma_P^{-1}(c)_{(-1)})_{(0)} \otimes \gamma_A(\gamma_P^{-1}(c)_{(-1)})_{(1)} \otimes \gamma_P^{-1}(c)_{(0)} \\ &= \gamma_A(\gamma_P^{-1}(c)_{(-2)}) \otimes \gamma_P^{-1}(c)_{(-1)} \otimes \gamma_P^{-1}(c)_{(0)} = (A \otimes {}^H\rho) \circ \gamma^{-1}_{A\square_H P}(c),\end{aligned}$$

where we used the H-colinearity of γ_A.

By Lemma 3.2.1, $A\square_H P$ is an algebra and a right C-comodule. For any $c \in C$,

$$\begin{aligned}&\gamma_{A\square_H P}(c_{(1)})\gamma^{-1}_{A\square_H P}(c_{(2)}) \\ &\quad = (\gamma_A(\gamma_P(c_{(1)})_{(-1)}) \otimes \gamma_P(c_{(1)})_{(0)}) \otimes (\gamma_A(\gamma_P^{-1}(c_{(2)})_{(-1)}) \otimes \gamma_P^{-1}(c_{(2)})_{(0)}) \\ &\quad = \gamma_A(\gamma_P(c_{(1)})_{(-1)})\gamma_A(\gamma_P^{-1}(c_{(2)})_{(-1)}) \otimes \gamma_P(c_{(1)})_{(0)}\gamma_P^{-1}(c_{(2)})_{(0)} \\ &\qquad\qquad \text{[use that } \gamma_A \text{ and } {}^H\rho \text{ are algebra maps]} \\ &\quad = \gamma_A((\gamma_P(c_{(1)})\gamma_P^{-1}(c_{(2)}))_{(-1)}) \otimes (\gamma_P(c_{(1)})\gamma_P^{-1}(c_{(2)}))_{(0)} \\ &\quad = \varepsilon(c)\gamma_A(1_{P(-1)}) \otimes 1_{P(0)} = \varepsilon(c)\gamma_A(1_H) \otimes 1_P = \varepsilon(c)1_A \otimes 1_P,\end{aligned}$$

and, similarly,

$$\begin{aligned}
&\gamma^{-1}_{A\Box_H P}(c_{(1)})\gamma_{A\Box_H P}(c_{(2)})\\
&\qquad = \gamma_A(\gamma_P^{-1}(c_{(1)})_{(-1)})\gamma_A(\gamma_P(c_{(2)})_{(-1)}) \otimes \gamma_P^{-1}(c_{(1)})_{(0)}\gamma_P(c_{(2)})_{(0)}\\
&\qquad\quad = \gamma_A((\gamma_P^{-1}(c_{(1)})\gamma_P(c_{(2)}))_{(-1)}) \otimes (\gamma_P^{-1}(c_{(1)})\gamma_P(c_{(2)}))_{(0)}\\
&\qquad\qquad = \varepsilon(c)1_A \otimes 1_P,
\end{aligned}$$

hence $\gamma^{-1}_{A\Box_H P}$ is the convolution inverse of $\gamma_{A\Box_H P}$ as claimed. Moreover, $\gamma_{A\Box_H P}$ is right colinear:

$$\begin{aligned}
\rho^C \circ \gamma_{A\Box_H P}(c) &= \gamma_A(\gamma_P(c)_{(-1)}) \otimes \gamma_P(c)_{(0)} \otimes \gamma_P(c)_{(1)}\\
&= \gamma_A(\gamma_P(c_{(1)})_{(-1)}) \otimes \gamma_P(c_{(1)})_{(0)} \otimes c_{(2)} = \gamma_{A\Box_H P}(c_{(1)}) \otimes c_{(2)}
\end{aligned}$$

Hence $\gamma_{A\Box_H P}$ is a cleaving map for $A\Box_H P$.

Assume now that $P \in \mathcal{M}_P^C(\psi)$ and that the entwining ψ commutes with the left H-coaction. Then, by Lemma 3.4.5, there exists an entwining $\psi_\Box$ such that $A\Box_H P \in \mathcal{M}^C_{A\Box_H P}(\psi_\Box)$. Hence, by Proposition 2.5.3, $(A\Box_H P)(R)^C_{\gamma_{A\Box_H P}}$ is a cleft C-coalgebra Galois extension. □

Chapter 4

Locally C-coalgebra Galois extensions

4.1 Introduction

Constructing new topological spaces by gluing together several known ones, or studying the properties of a given space by presenting it as a patching of topological spaces of a simpler structure, is the standard method in the classical topology which was frequently adapted to the noncommutative geometry. Examples include the quantum real projective sphere ([25]), and the Podleś sphere (defined in [35], it was proven in [40] and [25] that it is C^*-isomorphic with gluing of two quantum discs), the Matsumoto sphere and the quantum lens space ([30] and [31]).

The basic idea about covering quantum spaces by quantum subsets stems from the observation that an ideal of the algebra of functions on a given quantum space can be interpreted as consisting of those functions, which assume the value zero on some quantum subset. Then the quotient algebra can be viewed as the algebra of functions on this quantum subset.

Suppose that the algebra of functions on some quantum space has a family of ideals, which intersect at zero. Dually it means that the corresponding quantum subsets cover the whole of quantum space.

The covering and gluing of C^*-algebras and the notion of a locally trivial quantum principal bundles in the context of C^*-algebras were introduced in [14]. The purely algebraic theory of covering and gluing of algebras and differential algebras was presented in [15], which was followed by the purely algebraic definition of locally trivial quantum principal bundles in [16]. An example of a locally trivial principal bundle was produced in [16] and further elaborated in [24]. The theory resembling locally cleft extensions, based on sheaf theory was independently developed in [34].

In what follows we introduce the concept of *locally coalgebra Galois extensions* which, without going into technical details, are the algebras and comodules which have covers such that each of the quotient spaces are coalgebra-Galois extensions. Of a particular interest are the conditions which ensure that a locally coalgebra-Galois extension is a (global) coalgebra-Galois extension. Later we concentrate on the special case, in which all the quotient spaces of the cover of a locally Galois extension are cleft, which can be considered as the natural generalisation of locally trivial quantum principal bundles introduced in [16].

4.2 Covering of modules and algebras

In this and the next section we recall basic definitions and theorems from [15]. Note that the covering and gluing of modules was actually introduced in [17].

In what follows, all algebraic objects, unless specified otherwise, are $\mathbb{K}$-modules, where $\mathbb{K}$ is a unital commutative ring such that $\mathbb{K} \ni 2 \neq 0$, $\mathbb{K} \ni 3 \neq 0$, and any $\mathbb{K}$-module M considered is such that $2M = M$, $3M = M$.

Definition 4.2.1. Let A, B be algebras and let C be a coalgebra. Suppose that M is an (A, B)-bimodule (resp. a right C-comodule, an algebra, an algebra and a right C-comodule, etc.) Let I be a finite index set, and let $(J_i)_{i\in I}$ be a family of sub-bimodules (resp. C-sub-comodules, ideals, ideals which are also right C-sub-comodules, etc.) of M, such that

$$\bigcap_{i\in I} J_i = \{0\}. \tag{4.1}$$

Then the family $(J_i)_{i\in I}$ is called a *cover* or a *covering* of M.

In what follows we will only consider finite covers, i.e., in the statement '$(J_i)_{i\in I}$ is a covering' it should be implicitly understood that the index set I is finite.

Observe that the quotient modules

$$M_i = M/J_i, \quad M_{ij} = M/(J_i + J_j), \quad M_{ijk} = M/(J_i + J_j + J_k), \quad \ldots, \tag{4.2}$$

are (A, B)-bimodules (resp. C-comodules, algebras, algebras and right C-comodules, etc.), and hence, for all $i, j, k \in I$, the canonical surjections

$$\begin{aligned}
\pi_i : M \to M_i,\ m \mapsto m + J_i;\quad \pi_{ij} : M \to M_{ij},\ m \mapsto m + J_i + J_j;\\
\pi_{ijk} : M \to M_{ijk},\ m \mapsto m + J_i + J_j + J_k;\quad \ldots;\\
\pi^i_j : M_i \to M_{ij},\ m + J_i \mapsto m + J_i + J_j;\\
\pi^i_{jk} : M_i \to M_{ijk},\ m + J_i \mapsto m + J_i + J_j + J_k;\quad \ldots
\end{aligned} \tag{4.3}$$

are morphisms in the respective categories. Note that, for all $i, j, k \in I$, $M_{ii} = M_i = M_{iii}$, $M_{iij} = M_{ij}$, $M_{ij} = M_{ji}$, etc., and also

$$\pi_{ii} = \pi_i,\ \pi_{ij} = \pi_{ji},\ \pi^i_i = M_i,\ \pi^i_j \circ \pi_i = \pi_{ij},\ \pi^i_{jk} \circ \pi_i = \pi_{ijk},\ \text{etc.}, \tag{4.4}$$

A module

$$M^c = \{(m_i)_{i\in I} \in \bigoplus_{i\in I} M_i \mid \forall_{i,j\in I}\ \pi^i_j(m_i) = \pi^j_i(m_j)\} \tag{4.5}$$

is called a *covering completion* of M. Observe that $M^c = \ker \Psi_M$, where

$$\Psi_M : \bigoplus_{i\in I} M_i \to \bigoplus_{i,j\in I} M_{ij},\ (m_i)_{i\in I} \mapsto (\pi^i_j(m_i) - \pi^j_i(m_j))_{i,j\in I}. \tag{4.6}$$

The map

$$\kappa_M : M \to M^c, \ \ m \mapsto (\pi_i(m))_{i\in I} \tag{4.7}$$

is clearly injective (as $\ker \kappa_M = \bigcap_{i\in I} \ker \pi_i = \bigcap_{i\in I} J_i = \{0\}$). If κ_M is also surjective, then the cover $(J_i)_{i\in I}$ of M is called *a complete cover.*

Note that the definitions of the module M^c and the map κ_M make sense even if the family $(J_i)_{i\in I}$ is not a cover. Accordingly, in what follows, we shall make use of the term covering completion M^c and the map κ_M also when $\bigcap_{i\in I} J_i \neq \{0\}$. In fact, κ_M is injective if and only if $(J_i)_{i\in I}$ is a cover.

If M is an algebra with unit (and J_i, $i \in I$, are ideals), then M^c is an algebra, with unit $(1_{M_i})_{i\in I}$ and

$$(m_i)_{i\in I} \cdot (n_j)_{j\in I} = (m_i n_i)_{i\in I}, \ \text{ for all } (m_i)_{i\in I}, (n_j)_{j\in I} \in M^c. \tag{4.8}$$

The map κ_M is then an algebra morphism. Similarly, if C is a coalgebra, flat as a $\mathbb{K}$-module, and M is a C-comodule (and $(J_i)_{i\in I}$ is a family of sub-comodules), then M^c is naturally a C-comodule with the coaction

$$\rho^C : M^c \to M^c \otimes C, \ \ (m_i)_{i\in I} \mapsto (m_{i(0)} \otimes m_{i(1)})_{i\in I}. \tag{4.9}$$

With respect to this coaction κ_M is a right C-colinear map.

The following two propositions give criterions for a cover to be a complete one.

Lemma 4.2.2. *(Proposition 1, [15].) Let M be a $\mathbb{K}$-module, and let $J_1, J_2 \subseteq M$ be $\mathbb{K}$-submodules. Then the map $\kappa_M : M \to M^c$ defined by (4.7) is surjective. In particular, any covering by two subspaces is complete.*

Lemma 4.2.3. *(Proposition 3, [15].) Let M be a $\mathbb{K}$-module and let $(J_i)_{i\in I}$ be a covering of M. Assume that the index set is $I = \{1, 2, \ldots, n\}$ and that, for all $k \in I$, the submodules M_i satisfy*

$$\bigcap_{i\in\{1,2,\ldots,k-1\}} (J_i + J_k) = \left(\bigcap_{i\in\{1,2,\ldots,k-1\}} J_i \right) + J_k. \tag{4.10}$$

Then the covering $(J_i)_{i\in I}$ is complete.

The condition (4.10) is not necessary. Note however that the closed ideals of a C^*-algebra form a net with respect to intersection and addition, which is stronger condition than (4.10). Moreover, a similar but weaker than (4.10) condition is a necessary condition.

Lemma 4.2.4. *(Proposition 4, [15].) Let M be a $\mathbb{K}$-module, and let $(J_i)_{i\in I}$ be a complete covering of M. Then, for all $k \in I$,*

$$\bigcap_{i\neq k} (J_i + J_k) = \left(\bigcap_{i\neq k} J_i \right) + J_k. \tag{4.11}$$

4.3 Gluing of modules and algebras

Let M_i, M_{ij}, $i, j \in I$, be a finite family of modules, and let $\pi^i_j : M_i \to M_{ij}$ be a family of surjective homomorphisms such that $M_{ii} = M_i$, $M_{ij} = M_{ji}$ and $\pi^i_i = M_i$, for all $i, j \in I$. Then the module

$$\bigoplus_{\pi^i_j} M_i = \left\{ (m_i)_{i\in I} \in \bigoplus_{i\in I} M_i \mid \forall_{i,j\in I} \pi^i_j(m_i) = \pi^j_i(m_j) \right\} \tag{4.12}$$

is called a *gluing of the modules M_i with respect to π^i_j* (Definition 3 [15]).

Similarly as in the case of covering completions, if the modules M_i, M_{ij}, $i, j \in I$ are (unital) algebras and maps π^i_j are (unital) algebra maps, then gluing $\bigoplus_{\pi^i_j} M_i$ is an algebra. If C is a coalgebra flat as a $\mathbb{K}$-module, and modules M_i, M_{ij} are right C-comodules and the maps π^i_j are right C-colinear, then $\bigoplus_{\pi^i_j} M_i$ is naturally a right C-comodule.

Proposition 4.3.1. *(Proposition 8 [15].) Suppose that $M = \bigoplus_{\pi^i_j} M_i$. For all $i \in I$, let*

$$p_i : \bigoplus_{\pi^k_l} M_k \to M_i, \ (m_j)_{j\in I} \mapsto m_i. \tag{4.13}$$

Then $(\ker(p_i))_{i\in I}$ is a complete covering of M.

The maps p_i, $i \in I$, defined above are not in general surjective. The reason, given in [15], is that our definition of gluing (4.12) does not exclude self-gluing. The following proposition gives sufficient condition for surjectivity of maps p_i, $i \in I$.

Proposition 4.3.2. *(Proposition 9 [15].) Let $M = \bigoplus_{\pi^i_j} M_i$. Assume that the epimorphisms $\pi^i_j : M_i \to M_{ij}$ have the following properties.*

$$\text{For all } i, j, k \in I, \ \pi^i_j(\ker \pi^i_k) = \pi^j_i(\ker \pi^j_k). \tag{4.14}$$

Define isomorphisms

$$\begin{aligned} \theta^{ij}_k &: M_i/(\ker \pi^i_j + \ker \pi^i_k) \to M_{ij}/\pi^i_j(\ker \pi^i_k), \\ m_i + \ker \pi^i_j &+ \ker \pi^i_k \mapsto \pi^i_j(m_i) + \pi^i_j(\ker \pi^i_k). \end{aligned} \tag{4.15}$$

Then assume that the isomorphisms

$$\phi^k_{ij} = (\theta^{ij}_k)^{-1} \circ \theta^{ji}_k : M_j/(\ker \pi^j_i + \ker \pi^j_k) \to M_i/(\ker \pi^i_j + \ker \pi^i_k) \tag{4.16}$$

satisfy

$$\phi^j_{ik} = \phi^k_{ij} \circ \phi^i_{jk}, \text{ for all } i, j, k \in I. \tag{4.17}$$

Let $I = \{1, 2, \ldots, n\}$. If $n > 3$, assume that, for all $1 \le k < n$ and $1 \le i < k$,

$$\bigcap_{1\le j\le i} (\ker \pi_j^{k+1} + \ker \pi_{i+1}^{k+1}) = \left(\bigcap_{1\le j\le i} \ker \pi_j^{k+1} \right) + \ker \pi_{i+1}^{k+1}. \tag{4.18}$$

Then, for all $i \in I$, the maps p_i (4.13) are surjective.

Remark. Note that, for all $i, j, k \in I$, $m_j \in M_j$, $\phi_{ij}^k(m_j + \ker(\pi_i^j) + \ker(\pi_k^j)) = m_i + \ker(\pi_j^i) + \ker(\pi_k^i)$, where m_i is any element of M_i such that $\pi_j^i(m_i) = \pi_i^j(m_j)$.

4.4 Locally C-coalgebra Galois extensions

In what follows we shall frequently use the following two simple observations.

Lemma 4.4.1. *Suppose that A and B are algebras, and that $\pi : A \to B$ is a surjective algebra morphism. Take any $N \in {}_B\mathcal{M}_B$. Clearly $N \in {}_A\mathcal{M}_A$ with left and right A-actions defined by $a \cdot n \cdot a' = \pi(a)n\pi(a')$, for all $a, a' \in A$, $n \in N$. Then we can identify $N \otimes_B N$ with $N \otimes_A N$.*

Lemma 4.4.2. *Suppose that $P(B)^C$ is a C-coalgebra Galois extension. Let A be an algebra and a right C-comodule, and suppose that $\pi : P \to A$ is an algebra and a right C-comodule morphism. If $\pi(B) = A^{\mathrm{co}C}$, then $A(\pi(B))^C$ is a C-coalgebra Galois extension.*

Proof. It is clear that the map

$$\tau_A^C = (\pi \otimes_B \pi) \circ \tau_P^C : C \to A \otimes_B A \simeq A \otimes_{\pi(B)} A \tag{4.19}$$

is the translation map on A, where τ_P^C is the translation map on P, and we used the identification of $A \otimes_B A$ with $A \otimes_{\pi(B)} A$ (Lemma 4.4.1). □

Observe that it is not true in general that for an arbitrary coalgebra C and algebras and right C-comodules P and A, such that there exists an algebra surjection $\pi : P \to A$, we have $A^{\mathrm{co}C} = \pi(P^{\mathrm{co}C})$. As an example take C being a commutative Hopf algebra generated by a single primitive element x, i.e., $\Delta(x) = 1 \otimes x + x \otimes 1$ and $\varepsilon(x) = 0$. Let P be a free commutative algebra generated by two elements b and a, and let us define a right C-coaction $\rho^C : P \to P \otimes C$ as an algebra map defined by an algebra extension of the relations

$$\rho^C(b) = b \otimes 1, \ \ \rho^C(a) = a \otimes 1 + b \otimes x. \tag{4.20}$$

It is clear that $P^{\mathrm{co}C}$ is a subalgebra of P generated by the element b. Let $A = P/(Pb)$, i.e., A is a free commutative algebra generated by $a + Pb$, and let $\pi : P \to A$ be the canonical surjection on the quotient space. The map π is clearly right C-colinear, and moreover, C acts trivially on A, i.e, $A^{\mathrm{co}C} = A$. However, $\pi(P^{\mathrm{co}C}) = \mathbb{K}$.

Unfortunately P is not a C-Hopf Galois extension, and we were unable either to prove that $A^{\mathrm{co}C} = \pi(P^{\mathrm{co}C})$ when P is a C-coalgebra Galois extension, nor to produce a counterexample. The following lemma, however, shows that, under a not very restrictive condition, $A^{\mathrm{co}C} = \pi(P^{\mathrm{co}C})$ when P is a cleft extension.

Lemma 4.4.3. *Suppose that $P(B)^C$ is a cleft C-coalgebra Galois extension, and that $\pi : P \to A$ is a surjective algebra and right C-comodule morphism. Let $\gamma_P : C \to P$ be a cleaving map in P. If $\pi(1_{(0)}\gamma_P^{-1}(1_{(1)}))$ has a right inverse in $\pi(B)$, then $A(\pi(B))^C$ is a cleft C-coalgebra Galois extension.*

Proof. The map $\gamma_A = \pi \circ \gamma_P : C \to A$ is right C-colinear as the composition of two C-colinear maps, and it is convolution invertible, with the convolution inverse given by $\gamma_A^{-1} = \pi \circ \gamma_P^{-1}$. Moreover, since π is surjective, for all $a \in A$, there exists $p \in P$ such that $a = \pi(p)$, and then

$$a_{(0)}\gamma_A^{-1}(a_{(1)}) = \pi(p_{(0)}\gamma_P^{-1}(p_{(1)})) \in \pi(B) \subseteq A^{\mathrm{co}C}.$$

Therefore, by Proposition 2.5.3, it remains to prove that $A^{\mathrm{co}C} = \pi(B)$. Consider the map

$$B \otimes C \xrightarrow{b\otimes c \mapsto b\gamma_P(c)} P \xrightarrow{\pi} A \tag{4.21}$$

which is surjective by Proposition 2.5.3. Therefore, in particular, for all $s \in A^{coC}$, there exists $\sum_i b_i \otimes c_i \in B \otimes C$, such that $s = \pi(\sum_i b_i\gamma_P(c_i)) = \sum_i \pi(b_i)\gamma_A(c_i)$. The C-coinvariants of A are characterised by the property $\rho^C(s) = s\rho^C(1)$, therefore,

$$\sum_i \pi(b_i)\gamma_A(c_i)1_{(0)} \otimes 1_{(1)} = \sum_i \pi(b_i)\gamma_A(c_{i(1)}) \otimes c_{i(2)}.$$

Applying $m \circ (P \otimes \gamma_A^{-1})$ to both sides of the above equation yields

$$\sum_i \pi(b_i)\gamma_A(c_i)1_{(0)}\gamma_A^{-1}(1_{(1)}) = \sum_i \pi(b_i)\varepsilon(c_i),$$

hence, if $1_{(0)}\gamma_A^{-1}(1_{(1)})$ has a right inverse $R \in \pi(B)$, then

$$s = \sum_i \varepsilon(c_i)\pi(b_i)R \in \pi(B),$$

which ends the proof. □

Note that if $P(B)^C_{e,\gamma_P}$ is an e – *copointed* cleft C-coalgebra Galois extension, then $1_{(0)}\gamma_P^{-1}(1_{(1)}) = \gamma_P^{-1}(e)$, which is invertible in B with $(\gamma_P^{-1}(e))^{-1} = \gamma_P(e) \in B$.

Definition 4.4.4. A pair $(P(B)^C, (J_i)_{i\in I})$ is called a *locally C-coalgebra Galois extension* if the following conditions are satisfied.

1. P is an algebra and a right C-comodule and $B = P^{\mathrm{co}C}$.
2. The family $(J_i)_{i\in I}$ of ideals and right C-subcomodules of P is a complete cover of the algebra P.
3. For all $i \in I$, $\pi_i(B) = P_i^{\mathrm{co}C}$, and $P_i(\pi_i(B))^C$ is a C-coalgebra Galois extension.
4. For all $i, j \in I$, $\pi_{ij}(B) = P_{ij}^{\mathrm{co}C}$.

Note that while $(B\cap J_i)_{i\in I}$ is a cover of B it does not need to be a complete cover. Indeed, in general $B\cap J_i + B\cap J_j \neq B\cap (J_i+J_j)$, therefore $\bar{\pi}^i_j \neq \pi^i_j\big|_{B/(B\cap J_i)}$, where $\bar{\pi}^i_j : B/(B\cap J_i) \to B/(B\cap J_i + B\cap J_j)$, $b+B\cap J_i \mapsto b + B\cap J_i + B\cap J_j$.

The following lemma is very technical and apparently obvious. However, we shall make use of it several times in critical places and we want to state it explicitly.

Lemma 4.4.5. *Let I, J be index sets and suppose that C, M_i, $i\in I$, N_j, $j\in J$, are $\mathbb{K}$-modules. There is a well known canonical identification*

$$\vartheta_M : (\bigoplus_{i\in I} M_i)\otimes C \to \bigoplus_{i\in I}(M_i\otimes C),\ (m_i)_{i\in I}\otimes c \mapsto (m_i\otimes c)_{i\in I},$$

and similarly $\vartheta_N : (\bigoplus_{j\in J} N_j)\otimes C \simeq \bigoplus_{j\in J}(N_j\otimes C)$. Let $F^i_j : M_i \to N_j$, $i\in I, j\in J$ be a family of $\mathbb{K}$-linear morphisms. Define maps

$$F : \bigoplus_{i\in I} M_i \to \bigoplus_{j\in J} N_j,\ \ (m_i)_{i\in I} \mapsto (\sum_{i\in I} F^i_j(m_i))_{j\in J}, \tag{4.22}$$

$$G : \bigoplus_{i\in I}(M_i\otimes C) \to \bigoplus_{j\in J}(N_j\otimes C),\ (m_i\otimes c_i)_{i\in I} \mapsto (\sum_i F^i_j(m_i)\otimes c_i)_{j\in J}. \tag{4.23}$$

Then $G\circ\vartheta_M = \vartheta_N\circ(F\otimes C)$.

Lemma 4.4.6. *Suppose that $(P(B)^C, (J_i)_{i\in I})$ is a locally C-coalgebra Galois extension. For all $i\in I$, denote by $\tau_i : C\to P_i\otimes_B P_i$ the translation map in P_i. Then, for all $i,j\in I$,*

$$(\pi^i_j\otimes_B \pi^i_j)\circ\tau_i = (\pi^j_i\otimes_B \pi^j_i)\circ\tau_j. \tag{4.24}$$

Proof. By Lemma 4.4.2, both sides of (4.24) are translation maps in P_{ij}. But the translation map, if it exists, is unique, hence the equality. □

We use an indexed summation notation for the translation map. For all $i\in I$ and $c\in C$,

$$\tau_i(c) = c^{[1]_i}\otimes_B c^{[2]_i}, \tag{4.25}$$

implicit summation (not over i though!) is understood.

Proposition 4.4.7. *Let $(P(B)^C, (J_i)_{i\in I})$ be a locally C-coalgebra Galois extension and suppose that C is flat as a $\mathbb{K}$-module. Then P is a $(P,C)_\psi$ entwined module with*

$$\psi : C\otimes P \to P\otimes C,\ \ c\otimes p \mapsto (\kappa_P^{-1}\otimes C)((\psi_i(c\otimes\pi_i(p)))_{i\in I}), \tag{4.26}$$

where, for all $i\in I$, ψ_i is the canonical entwining on P_i (2.13).

Proof. First we prove that the map ψ is well defined. Using (4.24) we show that, for all $i,j\in I$, $c\in C$, $p\in P$,

$$(\pi^i_j\otimes C)\circ\psi_i(c\otimes\pi_i(p)) = (\pi^j_i\otimes C)\circ\psi_j(c\otimes\pi_j(p)). \tag{4.27}$$

Indeed,

$$
\begin{aligned}
(\pi^i_j \otimes C) \circ \psi_i(c \otimes \pi_i(p)) &= (\pi^i_j \otimes C)(c^{[1]_i}\rho^C(c^{[2]_i}\pi_i(p))) \\
&= \pi^i_j(c^{[1]_i})\rho^C(\pi^i_j(c^{[2]_i})\pi^i_j(\pi_i(p))) = \pi^j_i(c^{[1]_j})\rho^C(\pi^j_i(c^{[2]_j})\pi^j_i(\pi_j(p))) \\
&= (\pi^j_i \otimes C)(c^{[1]_j}\rho^C(c^{[2]_j}\pi_j(p))) = (\pi^j_i \otimes C) \circ \psi_j(c \otimes \pi_j(p)).
\end{aligned}
$$

Define the map

$$
\bar{\psi} : C \otimes P \to \bigoplus_{i \in I}(P_i \otimes C), \quad c \otimes p \mapsto (\psi_i(c \otimes \pi_i(p)))_{i \in I}.
$$

By (4.27) and Lemma 4.4.5, $\mathrm{Im}(\bar{\psi}) \subseteq \ker(\Psi_P \otimes C) = \ker(\Psi_P) \otimes C = P^c \otimes C$, where Ψ_P is as in (4.6), and we used the flatness of C and the definition of $P^c = \ker(\Psi_P)$. Hence the map $\psi = (\kappa_P^{-1} \otimes C) \circ \bar{\psi}$ is well defined.

In order to distinguish between different entwining maps we use indexed summation notation $\psi_i(c \otimes \pi_i(p)) = \pi_i(p)_{\alpha_i} \otimes c^{\alpha_i}$, for all $i \in I$, $p \in P$, $c \in C$, together with the usual notation $\psi(c \otimes p) = p_\alpha \otimes c^\alpha$, an implicit summation understood. We need to check whether ψ satisfies the bow-tie diagram condition (1.26). Indeed, for any $c \in C$,

$$
\psi(c \otimes 1_P) = (\kappa_P^{-1} \otimes C)((\psi_i(c \otimes 1_{P_i}))_{i \in I}) = (\kappa_P^{-1} \otimes C)((1_{P_i})_{i \in I} \otimes c) = 1_P \otimes c,
$$

where we used that κ_P (hence κ_P^{-1}) is a unital map. Similarly, for all $c \in C$, $p \in P$,

$$
\begin{aligned}
(P \otimes \varepsilon) \circ \psi(c \otimes p) &= (\kappa_P^{-1} \otimes \varepsilon)((\psi_i(c \otimes \pi_i(p)))_{i \in I}) \\
&= (\kappa_P^{-1} \otimes C)(((P_i \otimes \varepsilon) \circ \psi_i(c \otimes \pi_i(p)))_{i \in I}) = \kappa_P^{-1}((\pi_i(p))_{i \in I}\varepsilon(c)) = p\varepsilon(c).
\end{aligned}
$$

Observe that, for all $i \in I$,

$$
(\pi_i \otimes C) \circ \psi = \psi_i \circ (C \otimes \pi_i). \tag{4.28}
$$

Explicitly, for all $c \in C$, $p \in P$, $i \in I$, $\pi_i(p_\alpha) \otimes c^\alpha = \pi_i(p)_{\alpha_i} \otimes c^{\alpha_i}$. Hence, for all $c \in C$, $p, p' \in P$,

$$
\begin{aligned}
(pp')_\alpha \otimes c^\alpha &= (\kappa_P^{-1} \otimes C)((\pi_i(pp')_{\alpha_i} \otimes c^{\alpha_i})_{i \in I}) \\
&= (\kappa_P^{-1} \otimes C)((\pi_i(p)_{\alpha_i}\pi_i(p')_{\beta_i} \otimes c^{\alpha_i\beta_i})_{i \in I}) \\
&= (\kappa_P^{-1} \otimes C)((\pi_i(p_\alpha)\pi_i(p'_\beta) \otimes c^{\alpha\beta})_{i \in I}) = p_\alpha p'_\beta \otimes c^{\alpha\beta}.
\end{aligned}
$$

Similarly, for all $p \in P$, $c \in C$,

$$
\begin{aligned}
p_\alpha \otimes c^\alpha{}_{(1)} \otimes c^\alpha{}_{(2)} &= (\kappa_P^{-1} \otimes C \otimes C)((\pi_i(p_\alpha) \otimes c^\alpha{}_{(1)} \otimes c^\alpha{}_{(2)})_{i \in I}) \\
&= (\kappa_P^{-1} \otimes C \otimes C)((\pi_i(p)_{\alpha_i} \otimes c^{\alpha_i}{}_{(1)} \otimes c^{\alpha_i}{}_{(2)})_{i \in I}) \\
&= (\kappa_P^{-1} \otimes C \otimes C)((\pi_i(p)_{\alpha_i\beta_i} \otimes c_{(1)}{}^{\beta_i} \otimes c_{(2)}{}^{\alpha_i})_{i \in I}) \\
&= (\kappa_P^{-1} \otimes C \otimes C)((\pi_i(p_{\alpha\beta}) \otimes c_{(1)}{}^{\beta} \otimes c_{(2)}{}^{\alpha})_{i \in I}) = p_{\alpha\beta} \otimes c_{(1)}{}^{\beta} \otimes c_{(2)}{}^{\alpha}.
\end{aligned}
$$

It remains to prove that P is an entwined module. Indeed, using the coaction (4.9), for all $p, p' \in P$,

$$
\begin{aligned}
\rho^C(pp') &= \rho^C(\kappa_P^{-1} \circ \kappa_P(pp')) = (\kappa_P^{-1} \otimes C) \circ \rho^C \circ \kappa_P(pp') \\
&= (\kappa_P^{-1} \otimes C)((\rho^C(\pi_i(p)\pi_i(p')))_{i\in I}) = (\kappa_P^{-1} \otimes C)((\pi_i(p)_{(0)}\psi_i(\pi_i(p)_{(1)} \otimes \pi_i(p')))_{i\in I}) \\
&= (\kappa_P^{-1} \otimes C)((\pi_i(p_{(0)})\psi_i(p_{(1)} \otimes \pi_i(p')))_{i\in I}) = p_{(0)}(\kappa_P^{-1} \otimes C)((\psi_i(p_{(1)} \otimes \pi_i(p')))_{i\in I}) \\
&= p_{(0)}\psi(p_{(1)} \otimes p').
\end{aligned}
$$

□

Although a locally coalgebra Galois extension is built out of Galois extensions it is not necessarily a (global) coalgebra Galois extension. The next theorem, which is the main result of this section, gives (sufficient and necessary in the case of a flat coalgebra) conditions for when a locally coalgebra Galois extension is a global coalgebra Galois extension.

Theorem 4.4.8. *Let $(P(B)^C, (J_i)_{i\in I})$ be a locally C-coalgebra Galois extension.*

1. *If $(\ker(\pi_i \otimes_B \pi_i))_{i\in I}$ is a complete covering of $P \otimes_B P$ and*
$$\ker(\pi_{ij} \otimes_B \pi_{ij}) = \ker(\pi_i \otimes_B \pi_i) + \ker(\pi_j \otimes_B \pi_j), \tag{4.29}$$
then $P(B)^C$ is a C-coalgebra Galois extension.

2. *Suppose that the coalgebra C is flat as a $\mathbb{K}$ module. The family $(\ker(\pi_i \otimes_B \pi_i))_{i\in I}$ is a cover of $P \otimes_B P$ if and only if can_P^C is injective.*

3. *If $P(B)^C$ is a C-coalgebra Galois extension and C is flat as a $\mathbb{K}$-module, then the condition (4.29) is satisfied and $(\ker(\pi_i \otimes_B \pi_i))_{i\in I}$ is a complete covering of $P \otimes_B P$.*

Proof. Observe that, for all $i, j \in I$, modules $P_i \otimes_B P_i$ and $P_{ij} \otimes_B P_{ij}$, as the images of the surjective maps $\pi_i \otimes_B \pi_i$ and $\pi_{ij} \otimes_B \pi_{ij}$ respectively, can be identified with the respective quotient spaces $P \otimes_B P/\ker(\pi_i \otimes_B \pi_i)$ and $P \otimes_B P/\ker(\pi_{ij} \otimes_B \pi_{ij})$. Under this identification the maps $\pi_i \otimes_B \pi_i$ and $\pi_{ij} \otimes_B \pi_{ij}$ can be viewed as quotient maps.

1). Suppose first that $(\ker(\pi_i \otimes_B \pi_i))_{i\in I}$ is a complete cover of $P \otimes_B P$ and that relation (4.29) is satisfied. The condition (4.29) means that $P_{ij} \otimes_B P_{ij}$ can be identified with $P \otimes_B P/(\ker(\pi_i \otimes_B \pi_i) + \ker(\pi_j \otimes_B \pi_j))$. The map $\pi_j^i \otimes_B \pi_j^i$ is surjective and $(\pi_j^i \otimes_B \pi_j^i) \circ (\pi_i \otimes_B \pi_i) = \pi_{ij} \otimes_B \pi_{ij}$. Therefore, the map $\pi_j^i \otimes_B \pi_j^i$ can be viewed as a quotient map

$$
\begin{aligned}
&\pi_j^i \otimes_B \pi_j^i : P_i \otimes_B P_i \to P_{ij} \otimes_B P_{ij}, \\
&x + \ker(\pi_i \otimes_B \pi_i) \mapsto x + \ker(\pi_i \otimes_B \pi_i) + \ker(\pi_j \otimes_B \pi_j).
\end{aligned} \tag{4.30}
$$

It follows that the covering completion of $P \otimes_B P$ can be equivalently defined as

$$(P \otimes_B P)^c = \{(x_i)_{i\in I} \in \bigoplus_{i\in I} P_i \otimes_B P_i \mid \forall_{i,j\in I} (\pi_j^i \otimes_B \pi_j^i)(x_i) = (\pi_i^j \otimes_B \pi_i^j)(x_j)\}, \tag{4.31}$$

and then, by assumption, the map (cf. (4.7))

$$\kappa_{P\otimes_B P} : P \otimes_B P \mapsto (P \otimes_B P)^{\mathfrak{c}}, \;\; x \mapsto ((\pi_i \otimes_B \pi_i)(x))_{i\in I} \tag{4.32}$$

is bijective. Define a map

$$\tau^{\mathfrak{c}} : C \to (P \otimes_B P)^{\mathfrak{c}}, \;\; c \mapsto (\tau_i(c))_{i\in I}, \tag{4.33}$$

where $\tau_i : C \mapsto P_i \otimes_B P_i$ is the translation map on P_i, $i \in I$. Equation (4.24) ensures that this map has image in $(P \otimes_B P)^{\mathfrak{c}}$. We claim that the map

$$(\mathrm{can}_P^C)^{-1} : P \otimes C \to P \otimes_B P, \;\; p \otimes c \mapsto p\kappa^{-1}_{P\otimes_B P} \circ \tau^{\mathfrak{c}}(c) \tag{4.34}$$

is the inverse of the canonical map $\mathrm{can}_P^C : P \otimes_B P \to P \otimes C$ of P. Indeed, denote

$$\bar{\mathrm{can}} : (P \otimes_B P)^{\mathfrak{c}} \to \bigoplus_{i\in I} P_i \otimes C, \;\; (p_i \otimes_B q_i)_{i\in I} \mapsto (p_i q_{i(0)} \otimes q_{i(1)})_{i\in I}. \tag{4.35}$$

It is easy to see that

$$(\kappa_P \otimes C) \circ \mathrm{can}_P^C = \bar{\mathrm{can}} \circ \kappa_{P\otimes_B P}, \tag{4.36}$$

and therefore, for all $p \in P$ and $c \in C$,

$$\begin{aligned}
\mathrm{can}_P^C \circ (\mathrm{can}_P^C)^{-1}(p \otimes c) &= (\kappa_P^{-1} \otimes C) \circ \bar{\mathrm{can}} \circ \kappa_{P\otimes_B P}(p\kappa^{-1}_{P\otimes_B P} \circ \tau^{\mathfrak{c}}(c)) \\
&= (\kappa_P^{-1} \otimes C) \circ \bar{\mathrm{can}}((\pi_i(p)\tau_i(c))_{i\in I}) = (\kappa_P^{-1} \otimes C)((\pi_i(p) \otimes c)_{i\in I}) = p \otimes c.
\end{aligned}$$

Similarly, for all $p, q \in P$,

$$\begin{aligned}
(\mathrm{can}_P^C)^{-1} \circ \mathrm{can}_P^C(p \otimes_B q) &= (\mathrm{can}_P^C)^{-1}(pq_{(0)} \otimes q_{(1)}) = pq_{(0)}\kappa^{-1}_{P\otimes_B P} \circ \tau^{\mathfrak{c}}(q_{(1)}) \\
&= \kappa^{-1}_{P\otimes_B P}((\pi_i(p)\pi_i(q)_{(0)}\tau_i(\pi_i(q)_{(1)}))_{i\in I}) = \kappa^{-1}_{P\otimes_B P}(\pi_i(p) \otimes_B \pi_i(q)) = p \otimes_B q,
\end{aligned}$$

where in the fourth equality we used (2.8), and for the third equality we observed that $p\kappa^{-1}_{P\otimes_B P}((x_i)_{i\in I}) = \kappa^{-1}_{P\otimes_B P}((\pi_i(p)x_i)_{i\in I})$, for all $p \in P$, $(x_i)_{i\in I} \in (P \otimes_B P)^{\mathfrak{c}}$.
2.) Suppose that C is flat as a $\mathbb{K}$-module. It is clear that

$$\bar{\mathrm{can}}((P \otimes_B P)^{\mathfrak{c}}) \subseteq \ker(\Psi_P \otimes C) = P^{\mathfrak{c}} \otimes C,$$

where the last equality follows by the flatness of C and the definition of $P^{\mathfrak{c}}$ (4.31) and Ψ_P (4.6). Define

$$\mathrm{can}^{\mathfrak{c}} : (P \otimes_B P)^{\mathfrak{c}} \to P^{\mathfrak{c}} \otimes C, \;\; x \mapsto \bar{\mathrm{can}}(x). \tag{4.37}$$

It is clear that $\mathrm{can}^{\mathfrak{c}}$ is invertible with the inverse

$$(\mathrm{can}^{\mathfrak{c}})^{-1} : P^{\mathfrak{c}} \otimes C \to (P \otimes_B P)^{\mathfrak{c}}, \;\; (p_i)_{i\in I} \otimes c \to ((\mathrm{can}_{P_i}^C)^{-1}(p_i \otimes c))_{i\in I}. \tag{4.38}$$

Using (4.36), and noticing that $\kappa_P \otimes C$ is bijective, it is easy to see that $\kappa_{P\otimes_B P}$ is injective if and only if can_P^C is. But the injectivity of $\kappa_{P\otimes_B P}$ is equivalent to $(\ker(\pi_i \otimes_B \pi_i))_{i\in I}$ being a cover.

3. For brevity, we denote $\mathrm{can}_i = \mathrm{can}^C_{P_i}$, $\mathrm{can}_{ij} = \mathrm{can}^C_{P_{ij}}$. By Lemma 4.4.2, for all $i, j \in I$,

$$\mathrm{can}_i \circ (\pi_i \otimes_B \pi_i) = (\pi_i \otimes C) \circ \mathrm{can}^C_P, \tag{4.39}$$

$$\mathrm{can}_{ij} \circ (\pi_{ij} \otimes_B \pi_{ij}) = (\pi_{ij} \otimes C) \circ \mathrm{can}^C_P. \tag{4.40}$$

Hence, as the maps can^C_P, can_i, can_{ij} are bijective and C is flat as a $\mathbb{K}$-module, it follows that

$$\begin{aligned} \ker(\pi_{ij} \otimes_B \pi_{ij}) &= (\mathrm{can}^C_P)^{-1}(\ker(\pi_{ij} \otimes C)) = (\mathrm{can}^C_P)^{-1}(\ker(\pi_{ij}) \otimes C) \\ &= (\mathrm{can}^C_P)^{-1}(\ker(\pi_i) \otimes C + \ker(\pi_j) \otimes C) \\ &= (\mathrm{can}^C_P)^{-1}(\ker(\pi_i \otimes C)) + (\mathrm{can}^C_P)^{-1}(\ker(\pi_j \otimes C)) \\ &= \ker(\pi_i \otimes_B \pi_i) + \ker(\pi_j \otimes_B \pi_j). \end{aligned}$$

Furthermore, by (4.36), as the maps can^c, $\kappa_P \otimes C$ and can^C_P are bijective, we obtain that

$$\kappa_{P \otimes_B P} = (\mathrm{can}^c)^{-1} \circ (\kappa_P \otimes C) \circ \mathrm{can}^C_P$$

is invertible, hence the family $(\ker(\pi_i \otimes_B \pi_i))_{i \in I}$ is a complete cover of $P \otimes_B P$. □

Lemma 4.4.9. *Let $(P(B)^C, (J_i)_{i \in I})$ be a locally C-coalgebra Galois extension, and suppose that the ground ring $\mathbb{K}$ is a field. Then the condition (4.29) is satisfied.*

Proof. For all $i, j, k \in I$, the diagrams

$$\begin{tikzcd} 0 \arrow[r] & PdBP \arrow[r] \arrow[d, "\pi_i \otimes \pi_i|_{PdBP}"] & P \otimes P \arrow[r, "\Theta_P"] \arrow[d, "\pi_i \otimes \pi_i"] & P \otimes_B P \arrow[r] \arrow[d, "\pi_i \otimes_B \pi_i"] & 0 \\ 0 \arrow[r] & P_i dBP_i \arrow[r] & P_i \otimes P_i \arrow[r, "\Theta_{P_i}"] & P_i \otimes_B P_i \arrow[r] & 0 \end{tikzcd} \tag{4.41}$$

and

$$\begin{tikzcd} 0 \arrow[r] & PdBP \arrow[r] \arrow[d, "\pi_{ij} \otimes \pi_{ij}|_{PdBP}"] & P \otimes P \arrow[r, "\Theta_P"] \arrow[d, "\pi_{ij} \otimes \pi_{ij}"] & P \otimes_B P \arrow[r] \arrow[d, "\pi_{ij} \otimes_B \pi_{ij}"] & 0 \\ 0 \arrow[r] & P_{ij} dBP_{ij} \arrow[r] & P_{ij} \otimes P_{ij} \arrow[r, "\Theta_{P_{ij}}"] & P_{ij} \otimes_B P_{ij} \arrow[r] & 0, \end{tikzcd} \tag{4.42}$$

where $\Theta_M : M \otimes M \to M \otimes_B M$, $M \in {}_B\mathcal{M}_B$ is a natural surjection on the quotient space, and d denotes the universal differential, are clearly commutative and have exact rows.

Since the maps $\pi_i \otimes \pi_i|_{PdBP}$ and $\pi_{ij} \otimes \pi_{ij}|_{PdBP}$ are surjective, the application of the Snake Lemma to the above diagrams yields that the maps

$$\Theta_P|_{\ker(\pi_i \otimes \pi_i)} : \ker(\pi_i \otimes \pi_i) \to \ker(\pi_i \otimes_B \pi_i) \tag{4.43}$$

and

$$\Theta_P|_{\ker(\pi_{ij}\otimes\pi_{ij})} : \ker(\pi_{ij} \otimes \pi_{ij}) \to \ker(\pi_{ij} \otimes_B \pi_{ij}) \tag{4.44}$$

are well defined and surjective. Observe that as $\mathbb{K}$ is a field, for all $i, j \in I$,

$$\ker(\pi_i \otimes \pi_i) = \ker(\pi_i) \otimes P + P \otimes \ker(\pi_i),$$

and

$$\begin{aligned}\ker(\pi_{ij} \otimes \pi_{ij}) &= \ker(\pi_{ij}) \otimes P + P \otimes \ker(\pi_{ij}) \\ &= \ker(\pi_i) \otimes P + \ker(\pi_j) \otimes P + P \otimes \ker(\pi_i) + P \otimes \ker(\pi_j) \\ &= \ker(\pi_i \otimes \pi_i) + \ker(\pi_j \otimes \pi_j).\end{aligned}$$

Hence, for all $i, j \in I$,

$$\begin{aligned}\ker(\pi_{ij} \otimes_B \pi_{ij}) &= \Theta_P(\ker(\pi_{ij} \otimes \pi_{ij})) \\ &= \Theta_P(\ker(\pi_i \otimes \pi_i)) + \Theta_P(\ker(\pi_j \otimes \pi_j)) = \ker(\pi_i \otimes_B \pi_i) + \ker(\pi_j \otimes_B \pi_j).\end{aligned}$$

□

Therefore, Lemma 4.4.9 implies that, when working over a field, which is probably the most interesting case from a non-commutative geometry point of view, to verify whether a locally Galois extension is globally Galois, it suffices to check whether the covering $(\ker(\pi_i \otimes_B \pi_i))_{i\in I}$ is a complete covering of $P \otimes_B P$. More precisely we can state:

Corollary 4.4.10. *Suppose that the ground ring $\mathbb{K}$ is a field and that $(P(B)^C, (J_i)_{i\in I})$ is a locally C-coalgebra Galois extension. Then $P(B)^C$ is a C-coalgebra Galois extension if and only if $(\ker(\pi_i \otimes_B \pi_i))_{i\in I}$ is a complete cover of $P \otimes_B P$.*

In view of Corollary 4.4.10 it is important to study when a cover is a complete cover.

Lemma 4.4.11. *Let B be an algebra, and let $(K_i)_{i\in I}$ be a family of ideals of B. Denote the quotient spaces by $B_i = B/K_i$, $B_{ij} = B/(K_i + K_j)$, $i, j \in I$, and by*

$$\pi_i : B \to B_i, \quad \pi_{ij} : B \to B_{ij}, \quad \pi^i_j : B_i \to B_{ij}, \quad i, j \in I$$

the canonical surjections. Suppose that $M \in {}_B\mathcal{M}$, $M_i \in {}_{B_i}\mathcal{M}$, $M_{ij} \in {}_{B_{ij}}\mathcal{M}$, $i, j \in I$, is a family of modules such that $M_{ij} = M_{ji}$, for all $i, j \in I$, and that

$$\chi_i : M \to M_i, \quad \chi_{ij} : M \to M_{ij}, \quad \chi^i_j : M_i :\to M_{ij}, \quad i, j \in I,$$

is a family of surjective $\mathbb{K}$-linear morphisms with the properties

$$\ker(\chi_{ij}) = \ker(\chi_i) + \ker(\chi_j), \tag{4.45}$$

$$\chi^i_j \circ \chi_i = \chi_{ij} = \chi^j_i \circ \chi_j, \tag{4.46}$$

$$\chi_i(bm) = \pi_i(b)\chi_i(m), \quad \chi_{ij}(bm) = \pi_{ij}(b)\chi_{ij}(m), \tag{4.47}$$

for all $i, j \in I$, $b \in B$, $m \in M$.

Let $I = \{1, 2, \dots, n\}$, $n > 2$. Suppose that, for all $2 < k \leq n$,

$$\bigcap_{i \in \{1,2,\dots,k-1\}} \chi_k(\ker(\chi_i)) \subseteq \left(\prod_{i=1}^{k-1} \pi_k(K_i) \right) M_k. \tag{4.48}$$

Then, for all $2 < k \leq n$,

$$\bigcap_{i \in \{1,2,\dots,k-1\}} (\ker(\chi_i) + \ker(\chi_k)) = \left(\bigcap_{i \in \{1,2,\dots,k-1\}} \ker(\chi_i) \right) + \ker(\chi_k). \tag{4.49}$$

Proof. Note that by (4.46) and (4.45), for all $i, j \in I$, $\ker(\chi^i_j) = \chi_i(\ker(\chi_j))$. Therefore, for all $2 < k \leq n$,

$$\begin{aligned}
\bigcap_{i \in \{1,2,\dots,k-1\}} (\ker(\chi_i) + \ker(\chi_k)) &= \bigcap_{i \in \{1,2,\dots,k-1\}} \ker(\chi_{ik}) = \ker \left(\bigoplus_{i=1}^{k-1} \chi_{ik} \right) \\
&= \ker \left(\left(\bigoplus_{i=1}^{k-1} \chi^k_i \right) \circ \chi_k \right) = (\chi_k)^{-1} \left(\ker \left(\bigoplus_{i=1}^{k-1} \chi^k_i \right) \right) \\
&= (\chi_k)^{-1} \left(\bigcap_{i \in \{1,2,\dots,k-1\}} \ker(\chi^k_i) \right) = (\chi_k)^{-1} \left(\bigcap_{i \in \{1,2,\dots,k-1\}} \chi_k(\ker(\chi_i)) \right) \\
&\subseteq (\chi_k)^{-1} \left(\left(\prod_{i=1}^{k-1} \pi_k(K_i) \right) M_k \right) = \left(\prod_{i=1}^{k-1} K_i \right) M + \ker(\chi_k) \\
&\subseteq \left(\bigcap_{i \in \{1,2,\dots,k-1\}} \ker(\chi_i) \right) + \ker(\chi_k).
\end{aligned}$$

The inclusion relation in the opposite direction is always satisfied. □

Proposition 4.4.12. *Suppose that the ground ring $\mathbb{K}$ is a field. Let $(P(B)^C, (J_i)_{i \in I})$ be a locally C-coalgebra Galois extension, and let $\pi_i : P \to P_i = P/J_i$, etc., be the surjections on the quotient spaces. Let $K_i = B \cap J_i$, $i \in I$. Suppose that $I = \{1, 2, \dots, n\}$ and, for all $2 < k \leq n$,*

$$\bigcap_{i \in \{1,2,\dots,k-1\}} \pi_k(J_i) \subseteq \left(\prod_{i=1}^{k-1} \pi_k(K_i) \right) P_k. \tag{4.50}$$

Then if the family $(\ker(\pi_i \otimes_B \pi_i))_{i \in I}$ is a cover of $P \otimes_B P$, then it is a complete cover.

Proof. For all $i, j \in I$, denote $B_i = B/K_i$, $B_{ij} = B/K_{ij}$ and define maps

$$\chi_i = \mathrm{can}_i \circ (\pi_i \otimes_B \pi_i) : P \otimes_B P \to P_i \otimes C, \tag{4.51}$$
$$\chi_{ij} = \mathrm{can}_{ij} \circ (\pi_{ij} \otimes_B \pi_{ij}) : P \otimes_B P \to P_{ij} \otimes C, \tag{4.52}$$
$$\chi^i_j = \pi^i_j \otimes C : P_i \otimes C \to P_{ij} \otimes C, \tag{4.53}$$

which clearly satisfy the conditions (4.46)-(4.47). Moreover, as the maps can_i and can_{ij} are bijective, it follows that $\ker(\chi_i) = \ker(\pi_i \otimes_B \pi_i)$ and

$$\ker(\chi_{ij}) = \ker(\pi_{ij} \otimes_B \pi_{ij}) = \ker(\pi_i \otimes_B \pi_i) + \ker(\pi_j \otimes_B \pi_j) = \ker(\chi_i) + \ker(\chi_j),$$

where the second equality follows from Lemma 4.4.9. Moreover, for all $2 < k \leq n$,

$$\begin{aligned}\bigcap_{i\in\{1,2,\dots,k-1\}} \chi_k(\ker(\chi_i)) &= \bigcap_{i\in\{1,2,\dots,k-1\}} \ker(\pi_i^k \otimes C) = \bigcap_{i\in\{1,2,\dots,k-1\}} \ker(\pi_i^k) \otimes C \\ &= \ker\left(\bigoplus_{i=1}^{k-1} \pi_i^k\right) \otimes C = \left(\bigcap_{i\in\{1,2,\dots,k-1\}} \pi_k(J_i)\right) \otimes C \subseteq \left(\prod_{i=1}^{k-1} \pi_k(K_i)\right)(P_k \otimes C).\end{aligned}$$

Therefore, by Lemma 4.4.11, for all $2 < k \leq n$,

$$\begin{aligned}&\bigcap_{i\in\{1,2,\dots,k-1\}} (\ker(\pi_i \otimes_B \pi_i) + \ker(\pi_k \otimes_B \pi_k)) \\ &\qquad = \left(\bigcap_{i\in\{1,2,\dots,k-1\}} \ker(\pi_i \otimes_B \pi_i)\right) + \ker(\pi_k \otimes_B \pi_k),\end{aligned}$$

and hence, by Lemma 4.2.3, the family $(\ker(\pi_i \otimes_B \pi_i))_{i\in I}$ is a complete cover of $P \otimes_B P$. $\square$

The following lemma is probably well known, but we were unable to find the reference.

Lemma 4.4.13. *Let M, M', M'' be $\mathbb{K}$-modules and let $K, L \subseteq M$, $K', L' \subseteq M'$, $K'', L'' \subseteq M''$ be submodules. Suppose that*

$$f : K + L \to K' + L', \;\; g : K' + L' \to K'' + L''$$

are $\mathbb{K}$-linear maps such that the sequences

$$0 \longrightarrow K \xrightarrow{f|_K} K' \xrightarrow{g|_{K'}} K'' \longrightarrow 0, \tag{4.54}$$

$$0 \longrightarrow L \xrightarrow{f|_L} L' \xrightarrow{g|_{L'}} L'' \longrightarrow 0, \tag{4.55}$$

are well defined and exact. Then the sequence

$$0 \longrightarrow K \cap L \xrightarrow{f|_{K\cap L}} K' \cap L' \xrightarrow{g|_{K'\cap L'}} K'' \cap L'' \longrightarrow 0 \tag{4.56}$$

is exact if and only if the sequence

$$0 \longrightarrow K + L \xrightarrow{f} K' + L' \xrightarrow{g} K'' + L'' \longrightarrow 0 \tag{4.57}$$

is exact.

Proof. **(4.57)⇒(4.56).** Suppose that the sequence (4.57) is exact. Clearly $f|_{K\cap L}$ is injective and $g|_{K'\cap L'} \circ f|_{K\cap L} = 0$. Suppose that for some $m' \in K' \cap L'$, $g(m') = 0$. By the exactness of (4.54) and (4.55), there exist elements $k \in K$ and $l \in L$, such that $f(k) = m' = f(l)$, i.e., $f(k-l) = 0$. But f is, by the assumption, injective, therefore $k = l \in K \cap L$. Hence the sequence (4.56) is exact at $K' \cap L'$.

Finally, let $m'' \in K'' \cap L''$. By the exactness of (4.54) and (4.55), there exist elements $k \in K'$, $l \in L'$ such that $g(k') = m'' = g(l')$, hence $g(k'-l') = 0$. By the exactness of (4.57) at $K' + L'$, there exist elements $k \in K$, $l \in L$ such that $f(k+l) = k' - l'$, i.e., $k' - f(k) = l' + f(l) \in K' \cap L'$ and $g(k' - f(k)) = g(k') = m''$, hence $g(K' \cap L') = K'' \cap L''$.
(4.56)⇒(4.57). The map g is clearly surjective, as $g(K' + L') = g(K') + g(L') = K'' + L''$ by the exactness of (4.54) and (4.55). Similarly, for all $k \in K$, $l \in L$, $g \circ f(k+l) = g \circ f(k) + g \circ f(l) = 0$.

Suppose that, for some $k' \in K'$, $l' \in L'$, $g(k' + l') = 0$, i.e., $g(k') = g(-l') \in K'' \cap L''$. As, by the assumption $g(K' \cap L') = K'' \cap L''$, there exists $m' \in K' \cap L'$ such that $g(m') = g(k') = g(-l')$, i.e., $g(k' - m') = 0$ and $g(l' + m') = 0$. By the exactness of (4.54) and (4.55), there exists $k \in K, l \in L$, such that $f(k) = k' - m'$ and $f(l) = l' + m'$. Therefore $f(k+l) = k' - m' + l' + m' = k' + l'$, and we have proven that the sequence (4.57) is exact at $K' + L'$.

Finally, suppose that $f(k+l) = 0$, for some $k \in K$, $l \in L$, i.e, $f(k) = f(-l) \in K' \cap L'$. As $g(f(k)) = 0$, we have by the exactness of (4.56) at $K' \cap L'$ that there exists $m \in K \cap L$ such that $f(m) = f(k) = f(-l)$, i.e., $f(k-m) = 0$ and $f(l+m) = 0$. However $f|_K$ and $f|_L$ are by assumption injective, hence $k - m = 0$ and $l + m = 0$. Therefore $k + l = k - m + l + m = 0$ and we have proven that f is injective. □

Corollary 4.4.14. *We keep the notation and assumptions from the above lemma. Suppose that in addition we are given exact sequence of $\mathbb{K}$-maps*

$$0 \longrightarrow M \xrightarrow{s} M' \xrightarrow{t} M'' \longrightarrow 0, \tag{4.58}$$

such that $f = s|_{K+L}$ and $g = t|_{K'+L'}$. Then $g(K' \cap L') = K'' \cap L''$ if and only if $f(K + L) = \ker(g)$.

Proof. It is easy to see that, under the assumptions, the sequences (4.56) and (4.57) are exact, apart from the conditions $g(K' \cap L') = K'' \cap L''$ and $f(K + L) = \ker(g)$. □

Lemma 4.4.15. *Suppose that $\mathbb{K}$ is a field. Let $f : M \to N$, $g : M \to N'$ be $\mathbb{K}$-vector space morphisms such that* $\ker(f) \cap \ker(g) = \{0\}$. *Then*

$$\ker(f \otimes f) \cap \ker(g \otimes g) = \ker(f) \otimes \ker(g) + \ker(g) \otimes \ker(f). \tag{4.59}$$

Proof. As $\mathbb{K}$ is a vector space and $\ker(f) \cap \ker(g) = \{0\}$, we can write

$$M = \tilde{M} \oplus \ker(f) \oplus \ker(g),$$

and the assertion of the lemma easily follows. □

Proposition 4.4.16. *Suppose that $\mathbb{K}$ is a field and that $I = \{1,2\}$. Let $(P(B)^C, (J_i)_{i\in I})$ be a locally C-coalgebra Galois extension. Suppose that*

$$J_i = (B \cap J_i)P, \quad \text{for all } i \in I. \tag{4.60}$$

Then $P(B)^C$ is a C-coalgebra Galois extension if and only if

$$(\ker(\pi_1 \otimes \pi_1) + \ker(\pi_2 \otimes \pi_2)) \cap PdBP \\ = \ker(\pi_1 \otimes \pi_1) \cap PdBP + \ker(\pi_2 \otimes \pi_2) \cap PdBP.$$

Proof. By the Snake Lemma, for all $i, j \in I$, the commutative diagrams (4.41) and (4.42) with exact rows induce the exact sequences

$$0 \longrightarrow \ker(\pi_i \otimes \pi_i) \cap PdBP \longrightarrow \ker(\pi_i \otimes \pi_i) \longrightarrow \ker(\pi_i \otimes_B \pi_i) \longrightarrow 0 \tag{4.61}$$

and

$$0 \longrightarrow \ker(\pi_{ij} \otimes \pi_{ij}) \cap PdBP \longrightarrow \ker(\pi_{ij} \otimes \pi_{ij}) \longrightarrow \ker(\pi_{ij} \otimes_B \pi_{ij}) \longrightarrow 0. \tag{4.62}$$

By Lemma 4.4.9, the sequence (4.62) can be written as

$$0 \longrightarrow (\ker(\pi_i \otimes \pi_i) + \ker(\pi_j \otimes \pi_j)) \cap PdBP \longrightarrow \ker(\pi_i \otimes \pi_i) + \ker(\pi_j \otimes \pi_j) \\ \longrightarrow \ker(\pi_i \otimes_B \pi_i) + \ker(\pi_j \otimes_B \pi_j) \longrightarrow 0.$$

It follows immediately from Corollary 4.4.14 that

$$\Theta_P(\ker(\pi_i \otimes \pi_i) \cap \ker(\pi_j \otimes \pi_j)) = \ker(\pi_i \otimes_B \pi_i) \cap \ker(\pi_j \otimes_B \pi_j), \tag{4.63}$$

where $\Theta_P : P \otimes P \to P \otimes_B P$ is the natural surjection on the quotient space (cf. Lemma 4.4.9), if and only if

$$(\ker(\pi_i \otimes \pi_i) + \ker(\pi_j \otimes \pi_j)) \cap PdBP = \ker(\pi_i \otimes \pi_i) \cap PdBP + \ker(\pi_j \otimes \pi_j) \cap PdBP. \tag{4.64}$$

Let $I = \{1,2\}$. Hence $\ker(\pi_1) \cap \ker(\pi_2) = \{0\}$, and so, by Lemma 4.4.15,

$$\ker(\pi_1 \otimes \pi_1) \cap \ker(\pi_2 \otimes \pi_2) = \ker(\pi_1) \otimes \ker(\pi_2) + \ker(\pi_2) \otimes \ker(\pi_1). \tag{4.65}$$

Suppose that (4.60) is satisfied. Then by (4.65), $\Theta_P(\ker(\pi_i \otimes \pi_i) \cap \ker(\pi_j \otimes \pi_j)) = \Theta_P(\ker(\pi_1) \otimes \ker(\pi_2) + \ker(\pi_2) \otimes \ker(\pi_1)) = J_1 \otimes_B (J_2 \cap B)P + J_2 \otimes_B (J_1 \cap B)P = 0$ as $J_1 J_2, J_2 J_1 \subseteq J_1 \cap J_2 = \{0\}$. Therefore, if in addition (4.63) is satisfied, then $\ker(\pi_1 \otimes_B \pi_1) \cap \ker(\pi_2 \otimes_B \pi_2) = \{0\}$. □

4.5 Locally cleft extensions

Definition 4.5.1. A locally C-coalgebra Galois extension $(P(B)^C, (J_i)_{i\in I})$ is called a *locally cleft extension* if, for all $i \in I$, the quotient modules P_i are cleft C-coalgebra Galois extensions. It is called a *proper locally cleft extension* if, in addition, for all $i, j \in I$,

$$B \cap (J_i + J_j) = B \cap J_i + B \cap J_j. \tag{4.66}$$

We adopt the following notation. We denote $P_i = P/J_i$, $P_{ij} = P/(J_i + J_j)$, etc., as before. In addition, we have quotient modules $B_i = B/(B \cap J_i)$, $B_{ij} = B/((J_i \cap B) + (J_j \cap B))$, etc., for all $i, j \in I$. We reserve the use of the Greek letter π with various subscripts and superscripts to surjections onto the quotient modules of B, i.e., $\pi_i : B \to B_i$, $\pi^i_j : B_i \to B_{ij}$, $b + B \cap J_i \mapsto b + B \cap J_i + B \cap J_j$, etc. For quotient maps on P, we use sub- and superscripted letter χ, i.e., $\chi_i : P \to P_i$, etc.

For all $i \in I$, we denote by $\gamma_i : C \to P_i$ a cleaving map on P_i. Moreover, for all $i, j \in I$, we use the notation $\gamma^i_{ij} = \chi^i_j \circ \gamma_i : C \to P_{ij}$.

Note that, for all $i \in I$, $b \in B$, $\chi_i(b) = \pi_i(b)$ if we identify B_i with $B/J_i \subseteq P_i$. Similarly, we can naturally identify $B/(B \cap (J_i + J_j))$ with $B/(J_i + J_j) \subseteq P_{ij}$. In addition, if the relation (4.66) is satisfied, we can identify B_{ij} with $B/(J_i + J_j)$. Note that the condition (4.66) is equivalent to

$$\chi^i_j(b) = \pi^i_j(b), \text{ for all } i, j \in I,\ b \in B, \tag{4.67}$$

where we used the above identifications.

The condition (4.67) clearly implies that $(B \cap J_i)_{i \in I}$ is a complete cover of B.

Lemma 4.5.2. *(Cf. Lemma 1 [16].) Let $P(B)^C_\gamma$ be a cleft extension, and let J be an ideal in P such that $\rho^C(J) \subseteq J \otimes C$. Then there exists a left ideal K in B such that $J = K\gamma(C)$. Moreover, if the element $x = 1_{(0)}\gamma^{-1}(1_{(1)})$ has a right inverse y in P (i.e., $xy = 1_P$), and $Ky \subseteq K$, then K is a two-sided ideal and $K = J \cap B$.*

Proof. Let us define

$$K = (p \mapsto p_{(0)}\gamma^{-1}(p_{(1)}))(J). \tag{4.68}$$

Note that $K \subseteq J$. Therefore, $K\gamma(C) \subseteq J$. On the other hand, for all $p \in J$, $p = p_{(0)}\gamma^{-1}(p_{(1)})\gamma(p_{(2)}) \in K\gamma(C)$. Hence $J = K\gamma(C)$.

Let $b \in B$, $p \in J$, $b' = p_{(0)}\gamma^{-1}(p_{(1)}) \in K$. Then $bb' = bp_{(0)}\gamma^{-1}(p_{(1)}) = (bp)_{(0)}\gamma^{-1}((bp)_{(1)}) \in K$, hence K is a left ideal in B.

Suppose that the element $x = 1_{(0)}\gamma^{-1}(1_{(1)})$ has the right inverse y in P, and $Ky \subseteq K$. As shown above, $K \subseteq B \cap J$. On the other hand, let $b \in B \cap J$. Then $b = \sum_i k_i\gamma(c_i)$, for some $k_i \in K$, $c_i \in C$. It follows that

$$b1_{(0)} \otimes 1_{(1)} = \sum_i k_i\gamma(c_{i(1)}) \otimes c_{i(2)},$$

hence

$$b1_{(0)}\gamma^{-1}(1_{(1)}) = \sum_i k_i\gamma(c_{i(1)})\gamma^{-1}(c_{i(2)}) = \sum_i k_i\varepsilon(c_i),$$

and therefore $b = \sum_i k_i y\varepsilon(c_i) \in K$. Hence $K = J \cap B$ and it follows that K is a two-sided ideal in B. ☐

From the proof of the above lemma we immediately obtain

Corollary 4.5.3. *Suppose that $P(B)^C_\gamma$ is a cleft C-coalgebra Galois extension. Let K be an ideal in B, and let $J = K\gamma(C)$ be an ideal in P. Moreover suppose that the element $x = 1_{(0)}\gamma^{-1}(1_{(1)}) \in B$ has a right inverse in B. Then $K = J \cap B$.*

Definition 4.5.4. Let $(P(B)^C, (J_i)_{i\in I})$ be a locally cleft extension. Suppose that, for all $i \in I$, the element $1_{(0)}\gamma_i^{-1}(1_{(1)})$ has a right inverse in B_i. Such a locally cleft extension we shall call *a semiproper locally cleft extension.*

Suppose that $(P(B)^C, (J_i)_{i\in I})$ is a semiproper locally cleft extension. Then, by Lemma 4.4.3, for all $i, j, k \in I$, $P_{ij}^{\mathrm{co}C} = \chi_j^i(B_i) = \chi_{ij}(B)$, $P_{ijk}^{\mathrm{co}C} = \chi_{jk}^i(B_i) = \chi_{ijk}(B)$, etc.

For all $i, j \in I$, $\ker\chi_j^i$ is an ideal and a right C-subcomodule of P_i. Therefore, by Lemma 4.5.2, $\ker\chi_j^i = \bar{K}_j^i\gamma_i(C)$, where $\bar{K}_j^i = \ker(\chi_j^i) \cap B_i$ is an ideal in B_i. Note that $\ker\pi_j^i = \pi_i(\ker\pi_j) \subseteq \bar{K}_j^i$. Define $K_j^i = \pi_j^i(\bar{K}_j^i)$. Observe that

$$P_{ij}^{\mathrm{co}C} = \chi_j^i(B_i) = B_i/\bar{K}_j^i = \frac{B_i/\ker(\pi_j^i)}{\bar{K}_j^i/\ker(\pi_j^i)} = \pi_j^i(B_i)/\pi_j^i(\bar{K}_j^i) = B_{ij}/K_j^i. \tag{4.69}$$

A locally cleft extension $(P(B)^C, (J_i)_{i\in I})$ is proper if and only if, for all $i, j \in I$, $K_j^i = \{0\}$ (i.e., $\bar{K}_j^i = \ker\pi_j^i$). Note that the properness of a locally cleft extension implies that $B_{ij} = P_{ij}^{\mathrm{co}C}$, and then

$$\ker(\chi_{jk}^i) \cap B_i = \ker(\pi_{jk}^i), \;\; \ker(\chi_k^{ij}) \cap B_{ij} = \ker(\pi_k^{ij}), \;\; \text{for all } i, j, k \in I. \tag{4.70}$$

Indeed,

$$\begin{aligned}\ker(\chi_{jk}^i) &= \chi_i(\ker\chi_{jk}) = \chi_i(\ker\chi_j) + \chi_i(\ker\chi_k) = \ker(\chi_j^i) + \ker(\chi_k^i)\\ &= \ker(\pi_j^i)\gamma_i(C) + \ker(\pi_k^i)\gamma_i(C) = \ker(\pi_{jk}^i)\gamma_i(C),\end{aligned}$$

and

$$\ker(\chi_k^{ij}) = \chi_j^i \circ \chi_i(\ker\chi_k) = \chi_j^i(\ker(\pi_k^i)\gamma_i(C)) = \pi_j^i(\ker\pi_k^i)\gamma_{ij}^i(C) = \ker(\pi_k^{ij})\gamma_{ij}^i(C).$$

Then the relations (4.70) follow from Corollary 4.5.3. It follows that, for all $i, j, k \in I$, $B_{ijk} = \pi_k^{ij}(B_{ij})$ is isomorphic to $P_{ijk}^{\mathrm{co}C} = \chi_k^{ij}(B_{ij})$, and can be identified with it. Under this identification,

$$\chi_{jk}^i\Big|_{B_i} = \pi_{jk}^i, \;\; \chi_k^{ij}\Big|_{B_{ij}} = \pi_k^{ij}. \tag{4.71}$$

In what follows, we shall examine conditions for a semiproper locally cleft extension to be a proper locally cleft extension. This requires the study of ideals K_j^i, $i, j \in I$. We generalise steps of the proof of Proposition 2 [16].

For all $i, j \in I$, let us define an isomorphism (cf. (2.59))

$$\beta_{ij}^i : P_{ij} \to P_{ij}^{\mathrm{co}C} \otimes C, \;\; p_{ij} \mapsto \theta_{\gamma_{ij}^i}(p_{ij}) = p_{ij(0)}(\gamma_{ij}^i)^{-1}(p_{ij(1)}) \otimes p_{ij(2)}. \tag{4.72}$$

Lemma 4.5.5. *(Cf. the proof of Proposition 2 [16].) Suppose that $(P(B)^C, (J_i)_{i\in I})$ is a semiproper locally cleft extension. Then, for all $i, j \in I$, $K_j^i = K_i^j$.*

Proof. (Cf. the proof of Proposition 2 [16].) For all $i, j \in I$, define the map

$$\tilde{\phi}_{ji} : B_i \otimes C \to P_{ij}^{\mathrm{co}C} \otimes C, \ \ b_i \otimes c \mapsto \beta_{ij}^j \circ \chi_j^i(b_i\gamma_i(c)). \tag{4.73}$$

It is easy to see that $\ker \tilde{\phi}_{ji} = \bar{K}_j^i \otimes C$. Consider the maps

$$Q_{ji} : B_{ij} \to P_{ij}^{\mathrm{co}C}, \ \ \pi_j^i(b_i) \mapsto (P_{ij}^{\mathrm{co}C} \otimes \varepsilon) \circ \tilde{\phi}_{ji}(b_i 1_{(0)}\gamma_i^{-1}(1_{(1)}) \otimes 1_{(2)}). \tag{4.74}$$

Note that, as $\pi_i(\ker \pi_j) \subseteq \bar{K}_j^i$, and $\bar{K}_j^i$ is an ideal, for all $i, j \in I$, the maps Q_{ji} are well defined. Suppose that $b_{ij} \in K_j^i$. There exists an element $b_i \in \bar{K}_j^i$ such that $\pi_j^i(b_i) = b_{ij}$. It follows that $b_i 1_{(0)}\gamma_i^{-1}(1_{(1)}) \otimes 1_{(2)} \in \bar{K}_j^i \otimes C = \ker \tilde{\phi}_{ji}$, hence $Q_{ji}(b_{ij}) = 0$, and so $K_j^i \subseteq \ker Q_{ji}$, for all $i, j \in I$. On the other hand, for all $i, j \in I$ and $b \in B$,

$$\begin{aligned}
Q_{ji}(\pi_{ij}(b)) &= (P_{ij}^{\mathrm{co}C} \otimes \varepsilon) \circ \tilde{\phi}_{ji}(\pi_i(b)1_{(0)}\gamma_i^{-1}(1_{(1)}) \otimes 1_{(2)}) \\
&= (P_{ij}^{\mathrm{co}C} \otimes \varepsilon) \circ \beta_{ij}^j(\chi_{ij}(b)) = (P_{ij}^{\mathrm{co}C} \otimes \varepsilon)(\chi_{ij}(b_{(0)})(\gamma_{ij}^j)^{-1}(b_{(1)}) \otimes b_{(2)}) \\
&= \chi_i^j(\pi_j(b)1_{(0)}\gamma_j^{-1}(1_{(1)})) = \pi_{ij}(b)\pi_i^j(1_{(0)}\gamma_j^{-1}(1_{(1)})) + K_i^j.
\end{aligned}$$

Suppose that, for some element $b_{ij} \in B_{ij}$, $Q_{ji}(b_{ij}) = 0$, i.e., $b_{ij}\pi_j^i(1_{(0)}\gamma_j^{-1}(1_{(1)})) \in K_i^j$. But the element $1_{(0)}\gamma_j^{-1}(1_{(1)})$, by assumption, has a right inverse in B_j, and K_i^j is an ideal in B_{ij}, hence $b_{ij} \in K_i^j$. It follows that $\ker(Q_{ij}) = K_i^j$.

We have proven, that, for all $i, j \in I$, $K_j^i \subseteq K_i^j$, and therefore, for all $i, j \in I$, $K_j^i = K_i^j$. □

Let $(P(B)^C, (J_i)_{i \in I})$ be a semiproper locally cleft extension such that the coalgebra C is flat as a $\mathbb{K}$-module. Recall from the discussion following equation (4.7) that P^c is naturally an algebra and right C-comodule, and the map $\kappa_P : P \to P^c$, $p \mapsto (\chi_i(p))_{i \in I}$ is an algebra and a right C-comodule isomorphism. It follows that $\kappa_P(B) = (P^c)^{\mathrm{co}C}$. Is is clear that

$$\kappa_P(B) \subseteq \breve{B} = \{(b_i)_{i \in I} \in \bigoplus_{i \in I} B_i \mid \forall_{i,j \in I} \ \chi_j^i(b_i) = \chi_i^j(b_j)\}. \tag{4.75}$$

On the other hand, let $(b_i)_{i \in I} \in \breve{B}$, then, for all $(p_i)_{i \in I} \in P^c$,

$$\rho^C((b_i)_{i \in I}(p_j)_{j \in I}) = (\rho^C(b_i p_i))_{i \in I} = (b_i \rho^C(p_i))_{i \in I} = (b_i)_{i \in I}\rho^C((p_j)_{j \in I}),$$

i.e., $(b_i)_{i \in I} \in (P^c)^{\mathrm{co}C}$. It follows that $\breve{B} = (P^c)^{\mathrm{co}C} = \kappa_P(B)$.

Suppose that $(B \cap J_i)_{i \in I}$ is a complete covering of B. Then

$$\kappa_P(B) = B^c = \{(b_i)_{i \in I} \in \bigoplus_{i \in I} B_i \mid \forall_{i,j \in I} \ \pi_j^i(b_i) = \pi_i^j(b_j)\}. \tag{4.76}$$

Let $\mu_{ij} : B_{ij} \to B_{ij}/K^i_j = P^{\mathrm{co}C}_{ij}$ be the canonical surjections. Observe that $\chi^i_j\big|_{B_i} = \mu_{ij} \circ \pi^i_j$, for all $i, j \in I$. By (4.75) and (4.76), for all $(b_i)_{i\in I} \in \bigoplus_{i\in I} B_i$, the condition

$$\pi^i_j(b_i) = \pi^j_i(b_j), \text{ for all } i, j \in I, \tag{4.77}$$

is equivalent to

$$\mu_{ij}(\pi^i_j(b_i) - \pi^j_i(b_j)) = 0, \text{ for all } i, j \in I. \tag{4.78}$$

In particular, we have the following

Proposition 4.5.6. *(cf. Proposition 2, [15].) Let $(P(B)^C, (J_i)_{i\in I})$ be a semiproper locally cleft extension such that the coalgebra C is flat as a $\mathbb{K}$-module, and $I = \{1, 2\}$. Then $(P(B)^C, (J_i)_{i\in I})$ is a proper locally cleft extension.*

Proof. We prove by contradiction. Suppose that $K^1_2 \neq \{0\}$. Then there exists an element $r \in \bar{K}^1_2$ such that $\pi^1_2(r) \neq 0$. Let $(b_1, b_2) \in B^c$, then $(b_1 + r, b_2) \in \check{B}$ and $(b_1 + r, b_2) \notin B^c$ which, by the discussion preceding the above proposition, is impossible. □

Suppose that $(P(B)^C, (J_i)_{i\in I})$ is a proper and semiproper locally cleft extension. Let us define the family of gauge transformations

$$\Xi_{ij} : C \to B_{ij},\ c \mapsto \gamma^i_{ij}(c_{(1)})(\gamma^j_{ij})^{-1}(c_{(2)}), \text{ for all } i, j \in I. \tag{4.79}$$

The gauge transformations Ξ_{ij}, $i, j \in I$ satisfy the following conditions. For all $i, j, k \in I$, $c \in C$,

$$\Xi_{ii}(c) = \varepsilon(c), \quad \Xi_{ji} = (\Xi_{ij})^{-1}, \tag{4.80}$$

$$\pi^{ij}_k(\Xi_{ij}(c)) = \pi^{ik}_j(\Xi_{ik}(c_{(1)}))\pi^{kj}_i(\Xi_{kj}(c_{(2)})). \tag{4.81}$$

The first two of the above equalities are obvious, to prove the last one observe that, using (4.71), we obtain, for all $i, j, k \in I$,

$$\begin{aligned}
\pi^{ik}_j(\Xi_{ik}(c_{(1)}))\pi^{kj}_i(\Xi_{kj}(c_{(2)})) &= \chi^{ik}_j(\gamma^i_{ik}(c_{(1)})(\gamma^k_{ik})^{-1}(c_{(2)}))\chi^{kj}_i(\gamma^k_{kj}(c_{(3)})(\gamma^j_{kj})^{-1}(c_{(4)}))\\
&= \chi^i_{kj}(\gamma_i(c_{(1)}))\chi^k_{ij}(\gamma_k^{-1}(c_{(2)}))\chi^k_{ij}(\gamma_k(c_{(3)}))\chi^j_{ik}(\gamma_j^{-1}(c_{(4)}))\\
&= \chi^i_{kj}(\gamma_i(c_{(1)}))\chi^j_{ik}(\gamma_j^{-1}(c_{(2)})) = \chi^{ij}_k(\chi^i_j(\gamma_i(c_{(1)}))\chi^j_i(\gamma_j^{-1}(c_{(2)}))\\
&= \chi^{ij}_k(\gamma^i_{ij}(c_{(1)})(\gamma^j_{ij})^{-1}(c_{(2)})) = \pi^{ij}_k(\Xi_{ij}(c)).
\end{aligned}$$

Suppose that the ground ring $\mathbb{K}$ is a field and $I = \{1, 2\}$. Let $(P(B)^C, (J_i)_{i\in I})$ be a proper locally cleft C-coalgebra Galois extension. By Corollary 4.4.10, $P(B)^C$ is a C-coalgebra Galois extension if and only if

$$\ker(\chi_1 \otimes_B \chi_1) \cap \ker(\chi_2 \otimes_B \chi_2) = \{0\}. \tag{4.82}$$

We shall devote the remainder of the present section to proving that, under not very restrictive assumptions, the above condition is always satisfied.

Lemma 4.5.7. *Let M_1, M_2, M_{12} be $\mathbb{K}$-vector spaces, let $\pi_2^1 : M_1 \to M_{12}$, $\pi_1^2 : M_2 \to M_{12}$ be surjective linear morphisms. Let $M = \{(m,n) \in M_1 \oplus M_2 | \pi_2^1(m) = \pi_1^2(n)\}$, and denote the projections on the summands of the direct sum by $\pi_1 : M \to M_1$, $(m,n) \mapsto m$ and $\pi_2 : M \to M_2$, $(m,n) \mapsto n$. As $\mathbb{K}$ is a field, $\ker \pi_2^1$ and $\ker \pi_1^2$ are direct summands in M_1 and M_2 respectively, i.e., $M_1 = \overline{M}_1 \oplus \ker(\pi_2^1)$, $M_2 = \overline{M}_2 \oplus \ker(\pi_1^2)$, for some subspaces $\overline{M}_i \subseteq M_i$, $i = 1, 2$. Let*

$$\begin{array}{ll} \{m_i\} \text{ be a basis of } \ker \pi_2^1, & \{n_i\} \text{ be a basis of } \ker \pi_1^2, \\ \{\bar{m}_i\} \text{ be a basis of } \overline{M}_1, & \{\bar{n}_i\} \text{ be a basis of } \overline{M}_2. \end{array}$$

Suppose that $f : \overline{M}_1 \to M_2$ is a linear map such that, for all $m \in \overline{M}_1$, $\pi_2^1(m) = \pi_1^2(f(m))$. Then

the family $\{(0, n_i)\}$ is a basis of $\ker \pi_1$,
the family $\{(m_i, 0)\}$ is a basis of $\ker \pi_2$,
the family $\{(\bar{m}_i, f(\bar{m}_i))\}$ is linearly independent.

Moreover, denote $\overline{M} = Span(\{(\bar{m}_i, f(\bar{m}_i))\})$. Then $M = \overline{M} \oplus \ker(\pi_1) \oplus \ker(\pi_2)$.

Remark. Observe that the map f in Lemma 4.5.7 is necessarily injective. Note furthermore that the restriction $\bar{\pi}_2^1 : \overline{M}_1 \to M_{12}$ (resp. $\bar{\pi}_1^2 : \overline{M}_2 \to M_{12}$) of the map π_2^1 (resp. π_1^2) is an isomorphism, and, in particular, the map $f = (\bar{\pi}_1^2)^{-1} \circ \bar{\pi}_2^1 : \overline{M}_1 \to M_2$ satisfies the assumptions of the above lemma. Finally, the restriction $\bar{\pi}_{12} : \overline{M} \to M_{12}$ of the map $\pi_2^1 \circ \pi_1$ is an obvious linear isomorphism.

Proof. Suppose that $x \in \overline{M} \cap (\ker(\pi_1) + \ker(\pi_2))$. Then $x = (m, 0) + (0, n) = (m, n)$, for some elements $m \in \ker(\pi_2^1)$, $n \in \ker(\pi_1^2)$. On the other hand, $x \in \mathrm{Span}(\{(\bar{m}_i, f(\bar{m}_i))\})$, hence $x = \sum_i \alpha_i(\bar{m}_i, f(\bar{m}_i)) = (\sum_i \alpha_i \bar{m}_i, f(\sum_i \alpha_i \bar{m}_i))$, for some coefficients $\alpha_i \in \mathbb{K}$. Consequently, $\sum_i \alpha_i \bar{m}_i \in \ker(\pi_2^1)$. As $\mathrm{Span}(\{\bar{m}_i\}) \cap \ker(\pi_2^1) = \{0\}$, we have $x = 0$.

Suppose that $x \in M$. Then there exist unique coefficients $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{K}$, such that

$$\begin{aligned} x &= \left(\sum_i \alpha_i \bar{m}_i + \sum_j \beta_j m_j, \sum_k \gamma_k \bar{n}_k + \sum_l \delta_l n_l \right) \\ &= \sum_i \alpha_i (\bar{m}_i, f(\bar{m}_i)) + \left(0, \sum_k \gamma_k \bar{n}_k - f\left(\sum_i \alpha_i \bar{m}_i \right) \right) + \sum_j \beta_j (m_j, 0) + \sum_l \delta_l (0, n_l). \end{aligned}$$

Observe that

$$\pi_1^2 \left(\sum_k \gamma_k \bar{n}_k - f\left(\sum_i \alpha_i \bar{m}_i \right) \right) = \pi_1^2 \left(\sum_k \gamma_k \bar{n}_k \right) - \pi_2^1 \left(\sum_i \alpha_i \bar{m}_i \right) = 0.$$

Therefore there exist coefficients $\zeta_s \in \mathbb{K}$, such that $\sum_k \gamma_k \bar{n}_k - f(\sum_i \alpha_i \bar{m}_i) = \sum_s \zeta_s n_s$. It follows that M is spanned by vectors of the form $(m_i, 0)$, $(0, n_i)$, $(\bar{m}_i, f(\bar{m}_i))$. Their linear independence is obvious. $\square$

Lemma 4.5.8. *Assume that the ground ring $\mathbb{K}$ is a field, and suppose that $P(B)^C_\gamma$ is a cleft C-coalgebra Galois extension. Let $\{b_i\}$ be a linear basis of B and let $\{h_i\}$ be a linear basis of C. Then*

$$\{b_i\gamma(h_j) \otimes_B \gamma(h_k)\} \tag{4.83}$$

is a linear basis of $P \otimes_B P$.

Proof. The map

$$F : P \otimes_B P \to P \otimes C, \;\; p \otimes_B p' \mapsto pp'_{(0)}\gamma^{-1}(p'_{(1)}) \otimes p'_{(2)}, \tag{4.84}$$

is a linear isomorphism (cf. Proposition 2.5.3), such that, for all i, j, k,

$$F(b_i\gamma(h_j) \otimes_B \gamma(h_k)) = b_i\gamma(h_j) \otimes h_k. \tag{4.85}$$

Vectors $\{b_i\gamma(h_j) \otimes h_k\}$ form a basis of $P \otimes C$. We conclude that the family (4.83) is a basis of $P \otimes_B P$. □

Proposition 4.5.9. *Suppose that the ground ring $\mathbb{K}$ is a field and that $I = \{1,2\}$. Let $(P(B)^C, (J_i)_{i\in I})$ be a semiproper locally cleft C-coalgebra Galois extension. Assume that*

$$\gamma_2(C)\ker(\pi^2_1) = \ker(\pi^2_1)\gamma_2(C). \tag{4.86}$$

Then $P(B)^C$ is a C-coalgebra Galois extension.

Remark. Note that if C is a Hopf algebra and $P_2 = B_2 \otimes C$, then one can choose $\gamma_2 : C \to P_2$, $c \mapsto 1 \otimes c$, and then the condition (4.86) is automatically satisfied.

Proof. Suppose that $B_1 = \overline{B}_1 \oplus \ker(\pi^1_2)$, $B_2 = \overline{B}_2 \oplus \ker(\pi^2_1)$ and

$\{\bar{x}_i\}$ is a basis of $\overline{B}_1$, $\quad\{\bar{y}_i\}$ is a basis of $\overline{B}_2$, $\quad\{h_i\}$ is a basis of C,
$\{x_i\}$ is a basis of $\ker \pi^1_2$, $\quad\{y_i\}$ is a basis of $\ker \pi^2_1$.

By Proposition 2.5.3,

$$P_1 = \overline{B}_1\gamma_1(C) \oplus \ker(\pi^1_2)\gamma_1(C), \;\; P_2 = \overline{B}_2\gamma_2(C) \oplus \ker(\pi^2_1)\gamma_2(C).$$

Let $\overline{P}_1 = \overline{B}_1\gamma_1(C)$, $\overline{P}_2 = \overline{B}_2\gamma_2(C)$. By Proposition 4.5.6, $(P(B)^C, (J_i)_{i\in I})$ is a proper locally cleft extension. Therefore $\ker \chi^1_2 = \ker(\pi^1_2)\gamma_1(C)$ and $\ker \chi^2_1 = \ker(\pi^2_1)\gamma_2(C)$. It follows that

$$P_1 = \overline{P}_1 \oplus \ker(\chi^1_2), \;\; P_2 = \overline{P}_2 \oplus \ker(\chi^2_1), \tag{4.87}$$

and then, by Lemma 4.5.8,

$$\begin{array}{ll} \{\bar{x}_i\gamma_1(h_j)\} \text{ is a basis of } \overline{P}_1, & \{\bar{y}_i\gamma_2(h_j)\} \text{ is a basis of } \overline{P}_2 \\ \{x_i\gamma_1(h_j)\} \text{ is a basis of } \ker(\chi^1_2), & \{y_i\gamma_2(h_j)\} \text{ is a basis of } \ker(\chi^2_1). \end{array} \tag{4.88}$$

For all i, let $\check{x}_i$ denote the unique element of $\overline{B}_2$ such that $\pi^1_2(\bar{x}_i) = \pi^2_1(\check{x}_i)$. Similarily, for all i, let $\check{h}_i$ (resp. $\check{g}_i$) be the unique element of $\overline{P}_2$ (resp $\overline{P}_1$) with the property

$\chi_2^1(\gamma_1(h_i)) = \chi_1^2(\check{h}_i)$ (resp. $\chi_1^2(\gamma_2(h_i)) = \chi_2^1(\check{g}_i)$). Let us define the following elements of P:

$$\begin{array}{lll} X_i = \kappa_P^{-1}((x_i, 0)), & Y_j = \kappa_P^{-1}((0, y_j)), & \overline{X}_k = \kappa_P^{-1}((\bar{x}_k, \check{x}_k)), \\ H_s = \kappa_P^{-1}((\gamma_1(h_s), \check{h}_s)), & G_t = \kappa_P^{-1}((\check{g}_t, \gamma_2(h_t))), & \end{array} \tag{4.89}$$

for all i, j, k, s, t. Note that condition (4.86) implies that

$$\mathrm{Span}(\{Y_j G_t\}) = \mathrm{Span}(\{G_t Y_j\}). \tag{4.90}$$

Let us define $\overline{B} = \mathrm{Span}(\{\overline{X}_k\})$, $\overline{P} = \mathrm{Span}(\{\overline{X}_k H_s\})$. It follows immediately from Lemma 4.5.7 that

$$B = \overline{B} \oplus \ker(\pi_1) \oplus \ker(\pi_2), \quad P = \overline{P} \oplus \ker(\chi_1) \oplus \ker(\chi_2), \tag{4.91}$$

and

$$\begin{array}{llllll} \{X_i\} & \text{is a basis of} & \ker(\pi_2), & \{X_i H_s\} & \text{is a basis of} & \ker(\chi_2), \\ \{Y_j\} & \text{is a basis of} & \ker(\pi_1), & \{Y_j G_t\} & \text{is a basis of} & \ker(\chi_1), \\ \{\overline{X}_k\} & \text{is a basis of} & \overline{B}, & \{\overline{X}_k H_s\} & \text{is a basis of} & \overline{P}. \end{array} \tag{4.92}$$

Denote $\overline{P \otimes_B P} = \mathrm{Span}(\{\overline{X}_k H_s \otimes_B H_t\})$. We claim that

$$P \otimes_B P = \overline{P \otimes_B P} \oplus \ker(\chi_1 \otimes_B \chi_1) \oplus \ker(\chi_2 \otimes_B \chi_2), \tag{4.93}$$

and

$$\begin{array}{lll} \{\overline{X}_k H_s \otimes_B H_t\} & \text{is a basis of} & \overline{P \otimes_B P}, \\ \{X_i H_s \otimes_B H_t\} & \text{is a basis of} & \ker(\chi_2 \otimes_B \chi_2), \\ \{Y_j G_s \otimes_B G_t\} & \text{is a basis of} & \ker(\chi_1 \otimes_B \chi_1). \end{array} \tag{4.94}$$

First we prove that the above vectors span $P \otimes_B P$. Denote $(P \otimes_B P)' =$ (Span of the vectors (4.94)). By statements (4.92), it is obvious that the vectors

1)	$X_i H_s \otimes_B X_j H_t$,	2)	$X_i H_s \otimes_B Y_m G_t$,	3)	$X_i H_s \otimes_B \overline{X}_k H_t$,
4)	$Y_m G_s \otimes_B X_i H_t$,	5)	$Y_m G_s \otimes_B Y_n G_t$,	6)	$Y_m G_s \otimes_B \overline{X}_k H_t$,
7)	$\overline{X}_k H_s \otimes_B X_i H_t$,	8)	$\overline{X}_k H_s \otimes_B Y_n G_t$,	9)	$\overline{X}_k H_s \otimes_B \overline{X}_l H_t$,

for all i, j, k, l, m, n, s, t, span $P \otimes_B P$. Moving B-factors from the right to the left leg in each of the above tensor products, and then using the list (4.92), we obtain the following results.

The tensor products of types 2) and 4) are simply equal to zero, as

$$\ker(\chi_1)\ker(\pi_2) = \ker(\chi_2)\ker(\pi_1) = \{0\}.$$

The tensor products of types 1), 3), 7) clearly belong to

$$\ker(\chi_2) \otimes_B \mathrm{Span}(\{H_t\}) \subseteq \mathrm{Span}(\{X_i H_s \otimes_B H_t\}) \subseteq (P \otimes_B P)'.$$

The tensor products of types 5) and 8) clearly belong to

$$\ker(\chi_1) \otimes_B \mathrm{Span}(\{G_t\}) \subseteq \mathrm{Span}(\{Y_j G_s \otimes_B G_t\}) \subseteq (P \otimes_B P)'.$$

The tensor products of type 6) belong to

$$\begin{aligned}\ker(\chi_1)\otimes_B \mathrm{Span}(\{H_t\}) &= \mathrm{Span}(\{Y_jG_s\})\otimes_B \mathrm{Span}(\{H_t\})\\ &= \mathrm{Span}(\{G_sY_j\})\otimes_B \mathrm{Span}(\{H_t\}) \subseteq \mathrm{Span}(\{G_s\})\otimes_B \ker(\chi_1)\\ &= \mathrm{Span}(\{G_s\})\otimes_B \mathrm{Span}(\{Y_jG_t\})\\ &\subseteq \mathrm{Span}(\{Y_jG_s\otimes_B G_t\}) \subseteq (P\otimes_B P)',\end{aligned}$$

where in the second equality we used eq. (4.90).
The tensor products of type 9) belong to

$$\begin{aligned}P\otimes_B \mathrm{Span}(\{H_t\}) = {}&\mathrm{Span}(\{\overline{X}_kH_s\otimes_B H_t\})\oplus \mathrm{Span}(\{X_iH_s\otimes_B H_t\})\\ &\oplus \mathrm{Span}(\{Y_jG_s\otimes_B H_t\}).\end{aligned}$$

In the previous point, we have proven that the last direct summand in the above expression is also contained in $(P\otimes_B P)'$. Therefore tensors of type 9) belong to $(P\otimes_B P)'$. It follows that the tensors (4.94) span $P\otimes_B P$.

Suppose that the element $z\in\ker(\chi_1\otimes_B\chi_1)\cap\ker(\chi_2\otimes_B\chi_2)$. As tensors (4.94) span $P\otimes_B P$, there exists a family of coefficients $a_{ist}, b_{jst}, c_{kst}\in\mathbb{K}$, such that

$$z=\sum_{i,s,t} a_{ist}X_iH_s\otimes_B H_t+\sum_{j,s,t} b_{jst}Y_jG_s\otimes_B G_t+\sum_{k,s,t} c_{kst}\overline{X}_kH_s\otimes_B H_t. \tag{4.95}$$

Note that, for all i,j,k,s,

$$\begin{gathered}\chi_1(X_i)=x_i,\ \chi_1(Y_j)=0,\ \chi_1(\overline{X}_k)=\bar{x}_k,\ \chi_1(H_s)=\gamma_1(h_s),\\ \chi_2(Y_j)=y_j,\ \chi_2(G_s)=\gamma_2(h_s).\end{gathered}$$

It follows that

$$0=(\chi_1\otimes_B\chi_1)(z)=\sum_{i,s,t} a_{ist}x_i\gamma_1(h_s)\otimes_B\gamma_1(h_t)+\sum_{k,s,t} c_{kst}\bar{x}_k\gamma_1(h_s)\otimes_B\gamma_1(h_t).$$

By Lemma 4.5.8, this implies that $a_{ist}=c_{kst}=0$, for all i,k,s,t. Then

$$0=(\chi_2\otimes_B\chi_2)(z)=\sum_{j,s,t} b_{jst}y_j\gamma_2(h_s)\otimes_B\gamma_2(h_t).$$

It follows, by Lemma 4.5.8, that $b_{jst}=0$, for all j,s,t, i.e., $z=0$ and $\ker(\chi_1\otimes_B\chi_1)\cap\ker(\chi_2\otimes_B\chi_2)=\{0\}$. By Corollary 4.4.10, $P(B)^C$ is a C-coalgebra Galois extension. Note that we have also proven that the tensors (4.94) are linearly independent, and, consequently, they form a basis of $P\otimes_B P$. □

4.6 Gluing cleft extensions

The quantum geometry situation, which corresponds to the most usual setting for the classical method of constructing principal bundles by patching together trivial principal bundles, is as follows. We are given an algebra B, which has a complete covering $(K_i)_{i\in I}$, and a coalgebra C. For each of the quotient spaces $B_i = B/K_i$, $i \in I$ (resp. $B_{ij} = B/(K_i + K_j)$, $i,j \in I$), we construct a cleft C-coalgebra Galois extension $P_i(B_i)^C_{\gamma_i}$ (resp. $P_{ij}(B_{ij})^C_{\gamma_{ij}}$). Let us denote by $\pi_i : B \to B_i$, $\pi^i_j : B_i \to B_{ij}$, $i,j \in I$, etc., the canonical surjections. Then we choose surjective algebra and right C-comodule morphisms

$$\chi^i_j : P_i \to P_{ij}, \quad i,j \in I, \tag{4.96}$$

such that $\chi^i_j(b_i) = \pi^i_j(b_i)$, for all $i,j \in I$, $b_i \in B_i$, and we use them for gluing (cf. 4.12)

$$P = \bigoplus_{\chi^i_j} P_i = \{(p_i)_{i\in I} \in \bigoplus_{i\in I} P_i \mid \forall_{i,j\in I}\ \chi^i_j(p_i) = \chi^j_i(p_j)\}. \tag{4.97}$$

If the coalgebra C is flat as a $\mathbb{K}$-module then (cf. the discussion in Section 4.3) P is naturally a right C-comodule. Then, for each $n \in I$, we define the algebra and right C-comodule map

$$\chi_n : P \to P_n, \quad (p_i)_{i\in I} \mapsto p_n. \tag{4.98}$$

If all the maps χ_i, $i \in I$, are surjective then $(P(B)^C, (\ker \chi_i)_{i\in I})$ is a proper locally cleft C-coalgebra Galois extension. Moreover, for all $i \in I$, $P_i \simeq P/\ker(\chi_i)$. The following lemma gives necessary and sufficient conditions for the maps χ_i, $i \in I$, to be surjective.

Lemma 4.6.1. *Suppose that, for each $i \in I$, the element $1_{(0)}\gamma_i^{-1}(1_{(1)}) \in B_i$ has a right inverse in B_i. The algebra and right C-comodule maps (4.96) satisfy the condition $\chi^i_j(b_i) = \pi^i_j(b_i)$, for all $i,j \in I$, $b_i \in B_i$, if and only if there exists a family of convolution invertible maps (gauge transformations) $\Gamma^i_j : C \to B_{ij}$, $i,j \in I$, such that*

$$\chi^i_j(p_i) = \pi^i_j(p_{i(0)}\gamma_i^{-1}(p_{i(1)}))\Gamma^i_j(p_{i(2)})\gamma_{ij}(p_{i(3)}), \tag{4.99}$$

for all $i,j \in I$, $p_i \in P_i$. Furthermore, assume that the coalgebra C is flat as a $\mathbb{K}$-module. Let $I = \{1,2,\ldots,N\}$, and suppose that either $N \leq 3$, or the algebra B and its complete covering $(K_i)_{i\in I}$ satisfy the condition (cf. eq. (4.18)),

$$\bigcap_{1\leq j\leq i} (\ker \pi^{k+1}_j + \ker \pi^{k+1}_{i+1}) = \left(\bigcap_{1\leq j\leq i} \ker \pi^{k+1}_j\right) + \ker \pi^{k+1}_{i+1}, \tag{4.100}$$

for all $1 \leq k < N$ and $1 \leq i < k$. Define gauge transformations $\Xi_{ij} : C \to B_{ij}$, $c \mapsto \Gamma^i_j(c_{(1)})(\Gamma^j_i)^{-1}(c_{(2)})$, $i,j \in I$ (cf. eq. (4.79)). Then the maps (4.98) are surjective if and only if the condition (4.81) is satisfied:

$$\pi^{ij}_k(\Xi_{ij}(c)) = \pi^{ik}_j(\Xi_{ik}(c_{(1)}))\pi^{kj}_i(\Xi_{kj}(c_{(2)})). \tag{4.101}$$

Remark. It is clear that, while maps Γ^i_j, $i,j \in I$, define surjections χ^i_j, the space $P = \bigoplus_{\chi^i_j} P_i$ is fully defined by the maps Ξ_{ij}, $i,j \in I$. Indeed, for all $i,j \in I$, $p_i \in P_i$, $p_j \in P_j$, the condition $\chi^i_j(p_i) = \chi^j_i(p_j)$ is equivalent to

$$\pi^i_j(p_{i(0)}\gamma_i^{-1}(p_{i(1)}))\Xi_{ij}(p_{i(2)}) \otimes p_{i(3)} = \pi^j_i(p_{j(0)}\gamma_j^{-1}(p_{j(1)})) \otimes p_{j(2)}. \tag{4.102}$$

Proof. Suppose that the algebra and right C-comodule maps (4.96) satisfy the condition $\chi^i_j(b_i) = \pi^i_j(b_i)$, for all $i, j \in I$, $b_i \in B_i$. Then, for all $p_i \in P_i$,

$$\chi^i_j(p_i) = \chi^i_j(p_{i(0)}\gamma_i^{-1}(p_{i(1)})\gamma_i(p_{i(2)}))$$
$$= \pi^i_j(p_{i(0)}\gamma_i^{-1}(p_{i(1)}))\chi^i_j(\gamma_i(p_{i(2)}))\gamma_{ij}^{-1}(p_{i(3)})\gamma_{ij}(p_{i(4)})).$$

Defining the maps $\Gamma^i_j(c) = \chi^i_j(\gamma_i(c_{(1)}))\gamma_{ij}^{-1}(c_{(2)})$, $i, j \in I$ yields eq. (4.99). Conversely, let the maps (4.96) have the form (4.99). Then, for all $i, j \in I$, $b_i \in B_i$,

$$\chi^i_j(b_i) = \pi^i_j(b_i)\pi^i_j(1_{(0)}\gamma_i^{-1}(1_{(1)}))\Gamma^i_j(1_{(2)})\gamma_{ij}(1_{(3)}) = \pi^i_j(b_i)\chi^i_j(1) = \pi^i_j(b_i).$$

Assume that the coalgebra C is flat as a $\mathbb{K}$-module. We will check that the maps (4.96) satisfy the assumptions of Proposition 4.3.2, which in turn will prove that the maps (4.98) are surjective.

Note that

$$\ker \chi^i_j = \ker(\pi^i_j)\gamma_i(C), \text{ for all } i, j \in I. \tag{4.103}$$

Indeed, by (2.59), $\chi^i_j = \theta^{-1}_{\gamma^i_{ij}} \circ (\pi^i_j \otimes C) \circ \theta_{\gamma_i}$, which means that $\ker(\chi^i_j) = \theta^{-1}_{\gamma_i}(\ker(\pi^i_j \otimes C)) = \theta^{-1}_{\gamma_i}(\ker(\pi^i_j) \otimes C) = \ker(\pi^i_j)\gamma_i(C)$, where $\gamma^i_{ij} = \chi^i_j \circ \gamma_i$, and the second equality follows from the flatness of C. Observe that (condition (4.14) of Proposition 4.3.2)

$$\chi^i_j(\ker \chi^i_k) = \chi^j_i(\ker \chi^j_k), \text{ for all } i, j, k \in I. \tag{4.104}$$

Indeed, for all $i, j, k \in I$, $\pi^i_j(\ker \pi^i_k) = \pi^j_i(\ker \pi^j_k)$, and then, as $\mathbb{K}$-modules $\pi^i_j(\ker \pi^i_k)$ are ideals, for all $c \in C$, $\pi^i_j(\ker \pi^i_k)\Xi_{ij}(c_{(1)}) \otimes c_{(2)} \subseteq \pi^i_j(\ker \pi^i_k) \otimes C = \pi^j_i(\ker \pi^j_k) \otimes C$. Consequently, $\pi^i_j(\ker \pi^i_k)\gamma^i_{ij}(C) \subseteq \pi^j_i(\ker \pi^j_k)\gamma^j_{ij}(C)$. Furthermore by (4.103), $\chi^i_j(\ker \chi^i_k) = \pi^i_j(\ker \pi^i_k)\gamma^i_{ij}(C)$, hence it follows that, for all $i, j, k \in I$, $\chi^i_j(\ker \chi^i_k) \subseteq \chi^j_i(\ker \chi^j_k)$.

For each $i, j, k \in I$, the map

$$W^i_{jk} : P_i/(\ker(\chi^i_j) + \ker(\chi^i_k)) \to B_{ijk} \otimes C,$$
$$p_i + \ker(\chi^i_j) + \ker(\chi^i_k) \mapsto \pi^i_{jk}(p_{i(0)}\gamma_i^{-1}(p_{i(1)})) \otimes p_{i(2)}, \tag{4.105}$$

is an isomorphism. Indeed, since $\ker(\chi^i_j) + \ker(\chi^i_k) = \ker(\pi^i_{jk})\gamma_i(C)$, W^i_{jk} is well defined. It is also obviously surjective. Moreover, suppose that $W^i_{jk}(p_i + \ker(\chi^i_j) + \ker(\chi^i_k)) = 0$. This means that $\theta_{\gamma_i}(p_i) \in \ker(\pi^i_{jk} \otimes C) = \ker(\pi^i_{jk}) \otimes C$, hence $p_i \in \ker(\chi^i_j) + \ker(\chi^i_k)$. Note that, for all $b_i \in B_i$ and $c \in C$,

$$(W^i_{jk})^{-1}(\pi^i_{jk}(b_i) \otimes C) = b_i\gamma_i(c) + \ker(\chi^i_j) + \ker(\chi^i_k).$$

Suppose that the maps

$$\phi^k_{ij} : P_j/(\ker \chi^j_i + \ker \chi^j_k) \to P_i/(\ker \chi^i_j + \ker \chi^i_k), \quad i, j, k \in I$$

are the isomorphisms (4.16), i.e., for all $p_j \in P_j$,

$$\phi_{ij}^k(p_j + \ker(\chi_i^j) + \ker(\chi_k^j)) = p_i + \ker(\chi_j^i) + \ker(\chi_k^i),$$

where p_i is any element of P_i such that $\chi_j^i(p_i) = \chi_i^j(p_j)$ (cf. Remark after Proposition 4.3.2). For each $i, j, k \in I$, let us define the isomorphisms

$$\bar{\phi}_{ij}^k = W_{jk}^i \circ \phi_{ij}^k \circ (W_{ik}^j)^{-1} : B_{ijk} \otimes C \to B_{ijk} \otimes C. \tag{4.106}$$

It is easy to see that explicitly, for all $b_{ijk} \in B_{ijk}$, $c \in C$,

$$\bar{\phi}_{ij}^k(b_{ijk} \otimes c) = b_{ijk}\pi_k^{ij}(\Xi_{ji}(c_{(1)})) \otimes c_{(2)}. \tag{4.107}$$

Clearly, the condition (4.17) is equivalent to

$$\bar{\phi}_{ik}^j = \bar{\phi}_{ij}^k \circ \bar{\phi}_{jk}^i, \text{ for all } i, j, k \in I, \tag{4.108}$$

and this in turn is, by eq. (4.107), equivalent to the condition (4.101).

Finally, in view of the flatness of C and the condition (4.100) we obtain, for all $1 \leq k < N$ and $1 \leq i < k$,

$$\begin{aligned}
\bigcap_{1\leq j\leq i} (\ker \chi_j^{k+1} + \ker \chi_{i+1}^{k+1}) &= \theta_{\gamma_{k+1}}^{-1}\left(\bigcap_{1\leq j\leq i} \theta_{\gamma_{k+1}}(\ker \chi_j^{k+1} + \ker \chi_{i+1}^{k+1})\right) \\
&= \theta_{\gamma_{k+1}}^{-1}\left(\bigcap_{1\leq j\leq i} (\ker(\pi_{j,i+1}^{k+1}) \otimes C)\right) = \theta_{\gamma_{k+1}}^{-1}\left(\ker\left(\bigoplus_{1\leq j\leq i} \pi_{j,i+1}^{k+1} \otimes C\right)\right) \\
&= \theta_{\gamma_{k+1}}^{-1}\left(\ker\left(\bigoplus_{1\leq j\leq i} \pi_{j,i+1}^{k+1}\right) \otimes C\right) = \theta_{\gamma_{k+1}}^{-1}\left(\bigcap_{1\leq j\leq i} (\ker \pi_j^{k+1} + \ker \pi_{i+1}^{k+1}) \otimes C\right) \\
&= \theta_{\gamma_{k+1}}^{-1}\left(\bigcap_{1\leq j\leq i} (\ker \pi_j^{k+1}) \otimes C\right) + \ker \chi_{i+1}^{k+1} \\
&= \theta_{\gamma_{k+1}}^{-1}\left(\ker\left(\bigoplus_{1\leq j\leq i} \pi_j^{k+1} \otimes C\right)\right) + \ker \chi_{i+1}^{k+1} = \bigcap_{1\leq j\leq i} \ker(\chi_j^{k+1}) + \ker(\chi_{i+1}^{k+1}).
\end{aligned}$$

Thus all the assumptions of Proposition 4.3.2 are satisfied, and hence the maps (4.98) are surjective. □

4.7 Example: The quantum lens spaces

It was shown in [25] that by gluing two quantum discs, D_p and D_q, one can obtain the quantum 2-sphere S_{pq}^2, and that the universal C^* algebra of functions on S_{pq}

is isomorphic to equatorial or latitudinal Podleś spheres ([35]). In [16] a quantum sphere S^3_{pq} was obtained, by gluing quantum solid tori (cf. Subsection 2.5.1) $D_p \times S^1$ and $D_q \times S^1$, as an example of a locally trivial $U(1)$-quantum principal bundle with the base space S^2_{pq}. It was also shown that $\vartheta(S^3_{pq})(\vartheta(S^2_{pq}))^{\vartheta(U(1))}$ is a principal Hopf-Galois extension.

As an illustration of methods described earlier in this chapter, we will construct a locally cleft Hopf Galois extension of $\vartheta(S^2_{pq})$ by gluing two quantum solid tori (Subsection 2.5.1) $D_p \times_\theta S^1$ and $D_q \times_\theta S^1$, obtaining this way quantum lens spaces $L^{p,q,\theta}_\beta$ of charge β, for all $\beta \in \mathbb{Z}$. As a special case, for $\beta = 1$, this gives the Heegaard quantum sphere ([2]).

Another example of a construction of quantum lens spaces by gluing two quantum solid tori of type $D \times_\theta S^1$ can be found in [31].

4.7.1 Gluing of two quantum solid tori

Let $H = \vartheta(U(1))$ be the Hopf algebra generated by a unitary and group-like element u.

Let the deformation parameters $p, q \in (0,1)$, $\theta, \theta', \theta'' \in \mathbb{R}$. We define $P_1 = \vartheta(D_p \times_\theta S^1)$, $P_2 = \vartheta(D_q \times_{\theta'} S^1)$ (Subsection 2.5.1), $P_{12} = \vartheta(T_{\theta''})$ (Subsection 1.2.2). A $*$-algebra P_1 is generated by the elements x, h, which satisfy relations (2.64), and it is a right H-comodule algebra with the coaction defined by (2.68). The corresponding generators of P_2, y and g, satisfy the relations

$$\begin{aligned} gg^* = 1 = g^*g,\ \ y^*y - qyy^* = 1 - q, \\ gy = e^{i\theta'}yg,\ \ gy^* = e^{-i\theta'}y^*g. \end{aligned} \tag{4.109}$$

P_2 is a right H-comodule $*$-algebra with the coaction defined on generators by $\rho^H(y) = y \otimes 1$, $\rho^H(g) = g \otimes u$. Finally, P_{12} is a right H-comodule $*$-algebra generated by unitary elements U and V satisfying $UV = e^{i\theta''}VU$, with the right H-coaction defined by the relations $\rho^H(V) = V \otimes 1$, $\rho^H(U) = U \otimes u$. Note that $B_1 = P_1^{\mathrm{co}H} \simeq \vartheta(D_p)$ (see Section 1.2.1), is generated as a $*$-algebra by x, $B_2 = P_2^{\mathrm{co}H} \simeq \vartheta(D_q)$ is generated by y and $B_{12} = P_{12} \simeq \vartheta(S^1)$ is generated by V. Let the algebra surjections $\pi^1_2 : B_1 \to B_{12}$, $\pi^2_1 : B_2 \to B_{12}$ be defined on generators by

$$\pi^1_2(x) = V,\ \ \pi^2_1(y) = V. \tag{4.110}$$

We define cleaving maps by

$$\gamma_1^{\pm 1}(u^n) = h^{\pm n},\ \ \gamma_2^{\pm 1}(u^n) = g^{\pm n},\ \ \gamma_{12}^{\pm 1}(u^n) = U^{\pm n},\quad \text{for all } n \in \mathbb{Z}. \tag{4.111}$$

Then $P_1(B_1)^H_{\gamma_1}$, $P_2(B_2)^H_{\gamma_2}$, $P_{12}(B_{12})^H_{\gamma_{12}}$ are cleft extensions. By Lemma 4.6.1, in order to define gluing surjections (4.96), $\chi^1_2 : P_1 \to P_{12}$, $\chi^2_1 : P_2 \to P_{12}$, we need to find appropriate convolution invertible maps $\Gamma^1_2, \Gamma^2_1 : H \to B_{12}$. To fix the notation, without losing the generality, we shall only consider Γ^2_1.

For all $n \in \mathbb{Z}$, $\Gamma_1^2(u^n)$ and $(\Gamma_1^2)^{-1}(u^n)$ are Laurent polynomials in V such that $\Gamma_1^2(u^n)(\Gamma_1^2)^{-1}(u^n) = 1$. By the standard argument about degree counting, this implies that

$$(\Gamma_1^2)^{\pm 1}(u^n) = \mu(n)^{\pm 1} V^{\pm\nu(n)}, \text{ where } \mu : \mathbb{Z} \to \mathbb{C} \setminus \{0\},\ \nu : \mathbb{Z} \to \mathbb{Z}. \tag{4.112}$$

The map χ_1^2 must be algebraic, hence in paricular,

$$\chi_1^2(y^m g^n y^k g^l) = \chi_1^2(y^m g^n)\chi_1^2(y^k g^l), \text{ for all } m,n,k,l \in \mathbb{Z}.$$

Substituting (4.110), (4.111), (4.112) and (4.99) yields

$$\chi_1^2(y^m g^n y^k g^l) = \chi_1^2(e^{ink\theta'} y^m y^k g^{n+l}) = e^{ink\theta'} \mu(n+l) V^{m+k+\nu(n+l)} U^{n+l},$$

and

$$\begin{aligned} \chi_1^2(y^m g^n)\chi_1^2(y^k g^l) &= \mu(n)\mu(l) V^{m+\nu(n)} U^n V^{k+\nu(l)} U^l \\ &= e^{i\theta'' n(k+\nu(l))} \mu(n)\mu(l) V^{m+k+\nu(n)+\nu(l)} U^{n+l}. \end{aligned}$$

It follows that, for all $n, l, k \in \mathbb{Z}$,

$$\nu(n+l) = \nu(n) + \nu(l), \tag{4.113}$$

$$e^{ink(\theta'-\theta'')} \mu(n+l) = e^{i\theta'' n\nu(l)} \mu(n)\mu(l). \tag{4.114}$$

Condition (4.113) implies that, for all $n \in \mathbb{Z}$, $\nu(n) = \beta n$, where $\beta = \nu(1)$. Only left hand side of condition (4.114) depends on k, therefore it can be satisfied for all $k \in \mathbb{Z}$ only if $\theta' = \theta''$. We assume this, and then we have the following recursive relation

$$\mu(n+l) = e^{i\theta' n\beta l} \mu(n)\mu(l), \text{ for all } n,l \in \mathbb{Z}, \tag{4.115}$$

which has a family of solutions

$$\mu(n) = \alpha^n e^{i\beta\theta' \frac{n^2}{2}}, \text{ for all } n \in \mathbb{Z}, \tag{4.116}$$

where $\alpha \in \mathbb{C} \setminus \{0\}$. It follows that, for all $n \in \mathbb{Z}$, $\Gamma_1^2(u^n) = \alpha^n e^{i\beta\theta' \frac{n^2}{2}} V^{\beta n}$.

Similarly we prove that θ must equal θ'' and then, for all $n \in \mathbb{Z}$, $\Gamma_2^1(u^n) = (\alpha')^n e^{i\beta'\theta\frac{n^2}{2}} V^{\beta' n}$, for some $\alpha' \in \mathbb{C} \setminus \{0\}$ and $\beta' \in \mathbb{Z}$. In particular $\Gamma_2^1(u^n) = 1$, for all $n \in \mathbb{Z}$, is an admissible gauge transformation and, by the Remark after the Lemma 4.6.1, we can assume just that without losing any generality. Accordingly, the most general form of gluing maps for two quantum solid tori can be defined on the basis elements (cf. (2.65)) of respective solid tori as

$$\begin{aligned} \chi_2^1((1-xx^*)^k x^m h^n) &= \delta_{k0} V^m U^n, \\ \chi_1^2((1-yy^*)^k y^m g^n) &= \delta_{k0} \alpha^n e^{i\beta\theta\frac{n^2}{2}} V^{m+\beta n} U^n, \end{aligned} \tag{4.117}$$

for all $m, n \in \mathbb{Z}$, $k \in \mathbb{N}_0$. Note that χ_2^1 is a $*$-algebra map, and χ_1^2 is a $*$-algebra map if $|\alpha| = 1$. Observe that the glued algebra $P = \bigoplus_{\chi_j^i} P_i$ is a $*$-algebra in a natural way (i.e., with a $*$-operation defined by starring the components of the direct sum) if and only if the maps χ^i_j are $*$-algebra morphisms. On the other hand, scaling of g by a number of modulus one is an H-comodule $*$-algebra isomorphism of P_2. It follows that, if $|\alpha| = 1$, the parameter α can be absorbed, up to an isomorphism, by the redefinition $g \mapsto \alpha^{-1} g$. Accordingly, in what follows, we shall only consider the case $\alpha = 1$.

Let us denote the generators of the algebra $P_1^- = \vartheta(D_p \times_{-\theta} S^1)$ (resp. $P_2^- = \vartheta(D_q \times_{-\theta} S^1)$, $P_{12}^- = \vartheta(T_{-\theta})$) with the same symbols as the generators of P_1 (resp. P_2, P_{12}). We define, by the action on generators, the $*$-algebra isomorphisms

$$\begin{aligned} \eta_1 : P_1 \to P_1^-, &\ x \mapsto x,\ h \mapsto h^*, \\ \eta_2 : P_2 \to P_2^-, &\ y \mapsto y,\ g \mapsto g^*, \\ \eta_{12} : P_{12} \to P_{12}^-, &\ V \mapsto V,\ U \mapsto U^*. \end{aligned} \tag{4.118}$$

Clearly, for all $m, n \in \mathbb{Z}, k \in \mathbb{N}_0$,

$$\begin{aligned} \bar{\chi}^1_2 = \eta_{12} \circ \chi^1_2 \circ \eta_1^{-1} : P_1^- \to P_{12}^-, &\ (1 - xx^*)^k x^m h^n \mapsto \delta_{k0} V^m U^n, \\ \bar{\chi}^2_1 = \eta_{12} \circ \chi^2_1 \circ \eta_2^{-1} : P_2^- \to P_{12}^-, &\ (1 - xx^*)^k x^m h^n \mapsto \delta_{k0} e^{i\beta\theta\frac{n^2}{2}} V^{m-\beta n} U^n. \end{aligned} \tag{4.119}$$

Thus maps $\bar{\chi}^1_2$, $\bar{\chi}^2_1$ have the same form as maps (4.117) after substituting $\theta \mapsto -\theta$, $\beta \mapsto -\beta$. Denote $P^- = P_1^- \oplus_{\bar{\chi}^i_j} P_2^-$. It follows that the map

$$\eta = \eta_1 \oplus \eta_2 : P \to P^- \tag{4.120}$$

is a $*$-algebra isomorphism. Consequently, without losing generality, in what follows we shall confine ourselves to the case $\beta \geq 0$.

Using (4.117) and Lemma 4.5.7, it is easy to see, that the vectors

$$((1 - xx^*)^k x^m h^n, 0),\ \ (0, (1 - yy^*)^k y^m g^n),\ \ (x^m h^n, e^{-i\beta\theta\frac{n^2}{2}} y^{m-\beta n} g^n), \tag{4.121}$$

$m, n \in \mathbb{Z}, k > 0$, form a basis of P.

Lemma 4.7.1. *The elements*

$$\xi = (1 - xx^*, 0),\ z = (x, y),\ a = (e^{\frac{i\beta\theta}{2}} x^\beta h, g),\ b = (e^{\frac{i\beta\theta}{2}} h^{-1}, y^\beta g^{-1}) \tag{4.122}$$

of P generate P as a $*$-algebra.

Proof. It is enough to show that the basis vectors (4.121) are expressible in terms of elements (4.122). Observe that

$$(0, 1 - yy^*) = 1 - zz^* - \xi. \tag{4.123}$$

It follows immediately that, for all $k > 0$, $m, n \in \mathbb{Z}$,

$$((1 - xx^*)^k x^m h^n, 0) = e^{\frac{i\beta\theta}{2} n} \xi^k z^m b^{-n}, \tag{4.124}$$

and

$$(0,(1-yy^*)^k y^m g^n) = (1-zz^*-\xi)^k z^m a^n. \tag{4.125}$$

Furthermore, using equation (1.13), for all $m,n \in \mathbb{Z}$,

$$\begin{aligned}
&(x^m h^n, e^{-i\beta\theta\frac{n^2}{2}} y^{m-\beta n} g^n) \\
&\quad = (x^m h^n, e^{-i\beta\theta\frac{n^2}{2}} y^m y^{-\beta n} g^n) - (0, e^{-i\beta\theta\frac{n^2}{2}} y^{m-\beta n} Q^q_{m;-\beta n}(1-yy^*) g^n) \\
&\qquad = z^m (h^n, e^{-i\beta\theta\frac{n^2}{2}} y^{-\beta n} g^n) - e^{-i\beta\theta\frac{n^2}{2}} z^{m-\beta n} Q^q_{m;\beta n}(1-zz^*-\xi) a^n.
\end{aligned}$$

As $y^{-\beta n} g^n = e^{i\beta\theta\frac{n(n-1)}{2}} (y^{-\beta} g)^n$, it follows that

$$(h^n, e^{-i\beta\theta\frac{n^2}{2}} y^{-\beta n} g^n) = (h, e^{-\frac{i\beta\theta}{2}} y^{-\beta} g)^n = e^{\frac{i\beta\theta}{2} n} b^{-n}.$$

□

Lemma 4.7.2. *The generators ξ, z, a, b of P satisfy the following relations.*

$$\xi^* = \xi, \;\; \xi z = p z \xi, \;\; z^* z - q z z^* = 1 - q - (p-q)\xi, \tag{4.126a}$$
$$(1-zz^*-\xi)\xi = 0, \tag{4.126b}$$
$$\xi a = p^\beta a \xi, \;\; \xi b = b \xi, \;\; z a = e^{-i\theta} a z, \;\; z b = e^{i\theta} b z, \tag{4.126c}$$
$$z a^* - e^{i\theta} a^* z = (p^\beta - 1)\xi z^{1-\beta} b, \tag{4.126d}$$
$$z^* b - e^{-i\theta} b z^* = (1-q^\beta) z^{\beta-1} (1-zz^*-\xi) a^*, \tag{4.126e}$$
$$ab = e^{i\beta\theta} ba, \;\; ab^* = e^{-i\beta\theta} b^* a, \tag{4.126f}$$
$$ba = z^\beta, \tag{4.126g}$$
$$a^* a = \sum_{m=0}^{\beta} (-1)^m p^{\beta m - \frac{m(m-1)}{2}} \begin{bmatrix} \beta \\ m \end{bmatrix}_{p^{-1}} \xi^m, \tag{4.126h}$$
$$a a^* = \sum_{m=0}^{\beta} (-1)^m p^{-\beta m + \frac{m(m+1)}{2}} \begin{bmatrix} \beta \\ m \end{bmatrix}_{p} \xi^m, \tag{4.126i}$$
$$b^* b = \sum_{m=0}^{\beta} (-1)^m q^{\beta m - \frac{m(m-1)}{2}} \begin{bmatrix} \beta \\ m \end{bmatrix}_{q^{-1}} (1-zz^*-\xi)^m, \tag{4.126j}$$
$$b b^* = \sum_{m=0}^{\beta} (-1)^m q^{-\beta m + \frac{m(m+1)}{2}} \begin{bmatrix} \beta \\ m \end{bmatrix}_{q} (1-zz^*-\xi)^m. \tag{4.126k}$$

where $\begin{bmatrix} n \\ m \end{bmatrix}_p$ *are deformed binomial coefficients defined in (1.12)*

Proof. Easy if tedious proof is left to the reader. □

By the discussion in Section 4.3, the algebra P is naturally an H-comodule $*$-algebra. The coaction $\rho^H : P \to P \otimes H$ is defined on generators by

$$\rho^H(\xi) = \xi \otimes 1, \;\; \rho^H(z) = z \otimes 1, \;\; \rho^H(a) = a \otimes u, \;\; \rho^H(b) = b \otimes u^{-1}. \tag{4.127}$$

It is clear (cf. discussion around eq. (4.75)) that $P^{\mathrm{co}H} = B = B_1 \oplus_{\pi^i_j} B_2$. It follows that $P^{\mathrm{co}H}$ is generated by the elements $\xi, z \in P$.

4.7.2 Lens spaces of positive charge

Let $p, q \in (0,1)$, $\theta \in [0, 2\pi)$, $\beta \in \mathbb{N}_0$, and let $\vartheta(L_\beta^{p,q,\theta})$ be the quotient of a free $*$-algebra generated by the elements ξ, z, a, b, modulo the relations (4.126). We will call $\vartheta(L_\beta^{p,q,\theta})$ a *coordinate algebra of functions on a quantum lens space* $L_\beta^{p,q,\theta}$ *of positive charge* β.

Consider the family

$$\{\xi^k z^m b^n \mid k > 0,\ m,n \in \mathbb{Z}\},\ \ \{(1 - zz^* - \xi)^k z^m a^n \mid k \geq 0,\ m,n \in \mathbb{Z}\} \tag{4.128}$$

of vectors in $\vartheta(L_\beta^{p,q,\theta})$. We will prove that it is a basis of $\vartheta(L_\beta^{p,q,\theta})$. First we need to prove several technical lemmas. Let

$$\mathcal{A} = (\text{ Span of the family (4.128) }). \tag{4.129}$$

Lemma 4.7.3. *The elements* $1_{\mathcal{A}}$, ξ, z, z^*, a, a^*, b, b^* *belong to* $\mathcal{A}$.

Proof. The assertion is obvious in the case of the elements $1_{\mathcal{A}}$, ξ, z, z^*, a, a^*. Furthermore,

$$\begin{aligned} b = b\left(aa^* - \sum_{m=1}^{\beta} (-1)^m p^{-\beta m + \frac{m(m+1)}{2}} \begin{bmatrix} \beta \\ m \end{bmatrix}_p \xi^m\right) \\ = z^\beta a^* - \sum_{m=1}^{\beta} (-1)^m p^{-\beta m + \frac{m(m+1)}{2}} \begin{bmatrix} \beta \\ m \end{bmatrix}_p \xi^m b, \end{aligned}$$

where we used eq. (4.126i) in the first equality, and eq. (4.126g) in the second. Therefore $b \in \mathcal{A}$. Similarly, using eq. (4.126h) and eq. (4.126g), we obtain

$$b^* = e^{-i\beta\theta} z^{-\beta} a - \sum_{m=1}^{\beta} (-1)^m p^{\beta m - \frac{m(m-1)}{2}} \begin{bmatrix} \beta \\ m \end{bmatrix}_{p^{-1}} \xi^m b^* \in \mathcal{A}.$$

□

Lemma 4.7.4. *The following relations are satisfied in* $\vartheta(L_\beta^{p,q,\theta})$:

$$(1 - zz^* - \xi)z = qz(1 - zz^* - \xi), \tag{4.130a}$$

$$(1 - zz^* - \xi)a = a(1 - zz^* - \xi), \tag{4.130b}$$

$$(1 - zz^* - \xi)b = q^\beta b(1 - zz^* - \xi), \tag{4.130c}$$

$$\xi a^n = e^{i\beta\theta\frac{n(n+1)}{2}} \xi z^{\beta n} b^{-n},\ \textit{for all } n \in \mathbb{Z}. \tag{4.130d}$$

Proof. Using (4.126a) yields

$$\begin{aligned} (1 - zz^* - \xi)z &= z - z(z^*z) - pz\xi \\ &= z - z(1 - q(1 - zz^* - \xi) - p\xi) - pz\xi = qz(1 - zz^* - \xi). \end{aligned}$$

Furthermore,

$$\begin{aligned}(1-zz^*-\xi)a = a - z(z^*a) - p^\beta a\xi &= a - e^{i\theta}z(az^* + (1-p^\beta)b^*z^{\beta-1}\xi) - p^\beta a\xi \\ &= a - azz^* - e^{i\beta\theta}(1-p^\beta)z^\beta b^*\xi - p^\beta a\xi \\ &= a - azz^* - (1-p^\beta)abb^*\xi - p^\beta a\xi = a(1-zz^*-\xi),\end{aligned}$$

where in the forth equality we used eq. (4.126g). Similar proof of the equation (4.130c) is left to the reader as an exercise.

To prove the property (4.130d), we note that, by (4.130c), (4.126j) and (4.126k),

$$\xi b^n b^{-n} = 1, \text{ for all } n \in \mathbb{Z}. \tag{4.131}$$

Therefore, for all $n \in \mathbb{Z}$,

$$e^{i\beta\theta\frac{n(n+1)}{2}}\xi z^{\beta n}b^{-n} = e^{i\beta\theta\frac{n(n+1)}{2}}\xi(ba)^n b^{-n} = \xi a^n b^n b^{-n} = \xi a^n.$$

□

Lemma 4.7.5. *The vector subspace $\mathcal{A} \subseteq \vartheta(L_\beta^{p,q,\theta})$ (eq. (4.129)) is closed under multiplication.*

Proof. It is enough to consider products of basis vectors (4.128). Observe that, by equations (4.130a)–(4.130c),

$$\{\xi^k z^m b^n \mid k \in \mathbb{N},\ m,n \in \mathbb{Z}\} \cdot \{(1-zz^*-\xi)^l z^s a^t \mid l \in \mathbb{N},\ s,t \in \mathbb{Z}\} = \{0\},$$

and

$$\{(1-zz^*-\xi)^l z^s a^t \mid l \in \mathbb{N},\ s,t \in \mathbb{Z}\} \cdot \{\xi^k z^m b^n \mid k \in \mathbb{N},\ m,n \in \mathbb{Z}\} = \{0\}.$$

Note that $zz^* = 1 - (1-zz^*-\xi) - \xi$, and, by (4.126a),

$$z^*z = 1 - q(1-zz^*-\xi) - p\xi.$$

It follows, using (4.126a), (4.126b) and (4.130a), that, for all $n,m \in \mathbb{Z}$,

$$z^n z^m = (1 + \mathcal{P}_{n,m}(\xi) + \mathcal{Q}_{n,m}(1-zz^*-\xi))z^{n+m}, \tag{4.132a}$$

where $\mathcal{P}_{n,m}$ and $\mathcal{Q}_{n,m}$ are polynomials such that $\mathcal{P}_{n,m}(0) = \mathcal{Q}_{n,m}(0) = 0$. Similarly by relations (4.126h)–(4.126k), (4.126c) and (4.130b)–(4.130c), for all $n,m \in \mathbb{Z}$,

$$a^n a^m = (1 + \mathcal{P}'_{n,m}(\xi))a^{n+m}, \tag{4.132b}$$

$$b^n b^m = (1 + \mathcal{Q}'_{n,m}(1-zz^*-\xi))b^{n+m}, \tag{4.132c}$$

where polynomials $\mathcal{P}'_{n,m}$, $\mathcal{Q}'_{n,m}$, satisfy $\mathcal{P}'_{n,m}(0) = \mathcal{Q}'_{n,m}(0) = 0$.

It follows that, for all $m,n,s,t \in \mathbb{Z}$ and $k,l \in \mathbb{N}$,

$$\begin{aligned}(\xi^k z^m b^n)(\xi^l z^s b^t) = p^{-lm}e^{-ins\theta}\xi^{k+l}z^m z^s b^{n+t} &= p^{-lm}e^{-ins\theta}\xi^{k+l}(1+\mathcal{P}_{m,s}(\xi))z^{m+s}b^{n+t} \\ &\in \mathrm{Span}(\{\xi^k z^m b^n \mid k \in \mathbb{N}, m,n \in \mathbb{Z}\}) \subseteq \mathcal{A}.\end{aligned} \tag{4.133}$$

Using eq. (4.130d) yields, for all $m, n, s, t \in \mathbb{Z}, k \in \mathbb{N}$,

$$(\zeta^k z^m b^n)(z^s a^t) = e^{i\beta\theta\frac{t(t+1)}{2}} \zeta^k z^m b^n z^s z^{\beta t} b^{-t},$$

hence, by eq. (4.133),

$$(\zeta^k z^m b^n)(z^s a^t) \in \mathrm{Span}(\{\zeta^k z^m b^n \mid k \in \mathbb{N}, m, n \in \mathbb{Z}\}) \subseteq \mathcal{A}. \tag{4.134}$$

Analogously,

$$\begin{aligned}(z^s a^t)(\zeta^k z^m b^n) = p^{-\beta tk} z^s \zeta^k a^t z^m b^n &= e^{i\beta\theta\frac{t(t+1)}{2}} p^{-\beta tk} z^s \zeta^k z^{\beta t} b^{-t} z^m b^n \\ &\in \mathrm{Span}(\{\zeta^k z^m b^n \mid k \in \mathbb{N}, m, n \in \mathbb{Z}\}) \subseteq \mathcal{A}. \end{aligned} \tag{4.135}$$

In the remainder of the proof we need the following observation. For all $m, n \in \mathbb{Z}$,

$$a^n z^m \in e^{imn\theta} z^m a^n + \mathrm{Span}(\{\zeta^k z^m b^n \mid k \in \mathbb{N}, m, n \in \mathbb{Z}\}). \tag{4.136}$$

We use induction on $m, n \in \mathbb{Z}$. The above formula is obviously true for m or n equal to zero. By eq. (4.126c) and eq. (4.126d), it is also true for $m, n = \pm 1$. For brevity write $\mathcal{A}' = \mathrm{Span}(\{\zeta^k z^m b^n \mid k \in \mathbb{N}, m, n \in \mathbb{Z}\})$. Let $k = \pm 1$, $kn \geq 0$. Then, using equations (4.133), (4.134), (4.135), we obtain

$$\begin{aligned} a^{n+k} z^m = a^k(a^n z^m) &\in e^{imn\theta}(a^k z^m) a^n + a^k \mathcal{A}' \subseteq e^{imn\theta}(a^k z^m) a^n + \mathcal{A}' \\ &\subseteq e^{im(n+k)\theta} z^m a^{n+k} + e^{imn\theta} \mathcal{A}' a^n + \mathcal{A}' \subseteq e^{im(n+k)\theta} z^m a^{n+k} + \mathcal{A}'. \end{aligned}$$

Similarly, for $k = \pm 1$, $km \geq 0$,

$$\begin{aligned} a^n z^{m+k} = (a^n z^m) z^k &\in e^{imn\theta} z^m (a^n z^k) + \mathcal{A}' z^k \subseteq e^{imn\theta} z^m (e^{ink\theta} z^k a^n + \mathcal{A}') + \mathcal{A}' z^k \\ &\subseteq e^{i(m+k)n\theta} z^{m+k} a^n + \mathcal{A}'. \end{aligned}$$

Using (4.136) and then (4.132) and (4.130), we obtain, for all $m, n, s, t \in \mathbb{Z}$,

$$\begin{aligned} z^m a^n z^s a^t &\in e^{ins\theta} z^m z^s a^n a^t + z^m \mathcal{A}' a^t \\ &\subseteq e^{ins\theta}(1 + \mathcal{Q}_{m,s}(1 - zz^* - \zeta) + \mathcal{P}_{m,s}(\zeta)) z^{m+s} (1 + \mathcal{P}'_{n,t}(\zeta)) a^{n+t} + \mathcal{A}' \\ &= e^{ins\theta}(1 + \mathcal{Q}_{m,s}(1 - zz^* - \zeta)) z^{m+s} a^{n+t} + \mathcal{P}''(\zeta) z^{m+s+\beta(n+t)} b^{-(n+t)} + \mathcal{A}' \subseteq \mathcal{A}, \end{aligned}$$

where $\mathcal{P}''$ is a polynomial such that $\mathcal{P}''(0) = 0$, and in the last equality we used eq. (4.130d) and then eq. (4.132a). This shows that, for all $m, n, s, t \in \mathbb{Z}$ and $k, l \in \mathbb{N}_0$, $((1 - zz^* - \zeta)^k z^m a^n)((1 - zz^* - \zeta)^l z^s a^t) \in \mathcal{A}$, which ends the proof. □

Proposition 4.7.6. *Vectors (4.128) form a basis of $\vartheta(L_\beta^{p,q,\theta})$. The algebras $\vartheta(L_\beta^{p,q,\theta})$ and $P = P_1 \oplus_{\chi_j^i} P_2 = \vartheta(D_p \times_\theta S^1) \oplus_{\chi_j^i} \vartheta(D_q \times_\theta S^1)$ are mutually isomorphic. Here the maps χ_2^1 and χ_1^2 are defined in eq. (4.117) with $\alpha = 1$ and $\beta \geq 0$.*

Proof. Let the algebra maps $\chi_i : \vartheta(L_\beta^{p,q,\theta}) \to P_i$, $i = 1, 2$, be defined on generators by

$$\chi_1(\xi) = 1 - xx^*,\ \chi_1(z) = x,\ \chi_1(a) = e^{\frac{i\beta\theta}{2}} x^\beta h,\ \chi^1(b) = e^{\frac{i\beta\theta}{2}} h^{-1},$$
$$\chi_2(\xi) = 0,\ \chi_2(z) = y,\ \chi_2(a) = g,\ \chi_2(b) = y^\beta g^{-1}. \tag{4.137}$$

By Lemmas 4.7.1 and 4.7.2 these maps are well defined, and by Lemma 4.7.1, the map $\chi = \chi_1 \oplus \chi_2 : \vartheta(L_\beta^{p,q,\theta}) \to P$ is surjective. Let $w \in \ker\chi$. By Lemmas 4.7.3 and 4.7.5, vectors (4.128) span $\vartheta(L_\beta^{p,q,\theta})$, hence

$$w = \sum_{\substack{m,n\in\mathbb{Z}\\ k>0}} \mu_{kmn}\xi^k z^m b^n + \sum_{\substack{s,t\in\mathbb{Z}\\ l\geq 0}} \nu_{lst}(1 - zz^* - \xi)^l z^s a^t, \tag{4.138}$$

for some coefficients $\mu_{kmn}, \nu_{lst} \in \mathbb{C}$, where $m, n, s, t \in \mathbb{Z}$, $k > 0$, $l \geq 0$. By assumption, $\chi_1(w) = 0$ and $\chi_2(w) = 0$. It follows that

$$0 = \chi_2(w) = \sum_{\substack{s,t\in\mathbb{Z}\\ l\geq 0}} \nu_{lst}(1 - yy^*)^l y^s g^t.$$

Since the elements $(1 - yy^*)^l y^s g^t$, $l \in \mathbb{N}_0$, $s, t \in \mathbb{Z}$, form a linear basis of P_2, this implies that, for all $l \in \mathbb{N}_0$, $s, t \in \mathbb{Z}$, $\nu_{lst} = 0$. But then

$$0 = \chi_1(w) = \sum_{\substack{m,n\in\mathbb{Z}\\ k>0}} \mu_{kmn} e^{\frac{i\beta\theta}{2}n}(1 - xx^*)^k x^m h^n,$$

which implies that, for all $k \in \mathbb{N}$ and $m, n \in \mathbb{Z}$, $\mu_{kmn} e^{\frac{i\beta\theta}{2}n} = 0$ and so $\mu_{kmn} = 0$. Hence $w = 0$ and therefore $\ker\chi = \{0\}$ and so $\chi : \vartheta(L_\beta^{p,q,\theta}) \to P$ is a $*$-algebra isomorphism. It follows that we can identify $\vartheta(L_\beta^{p,q,\theta})$ with P. We have also proven that vectors (4.128) are linearly independent and hence they form a linear basis of $\vartheta(L_\beta^{p,q,\theta})$.

Let us define a right H-coaction $\rho^H : \vartheta(L_\beta^{p,q,\theta}) \to \vartheta(L_\beta^{p,q,\theta}) \otimes H$ by (eq. 4.127), which makes $\chi : \vartheta(L_\beta^{p,q,\theta}) \to P$ an H-comodule isomorphism. It follows, by the discussion after eq. (4.127), that $B = P^{\mathrm{co}H}$ is isomorphic to the quotient of a free algebra, generated by elements, ξ, z, by the relations (4.126a)-(4.126b). This, in turn, is the coordinate algebra $\vartheta(S^2_{pq})$ on the quantum 2-sphere S^2_{pq} defined, by gluing two quantum discs D_p and D_q, in [25]. □

4.7.3 The inverse of the canonical map on $\vartheta(L_\beta^{p,q,\theta})$

By Proposition 4.5.9, $P(B)^H$ is an H-Hopf Galois extension. The translation maps on P_1 and P_2 are given explicitly, by the formulae, for all $n \in \mathbb{Z}$,

$$\tau_1 : H \to P_1 \otimes_B P_1,\ u^n \mapsto h^{-n} \otimes_B h^n,$$
$$\tau_2 : H \to P_2 \otimes_B P_2,\ u^n \mapsto g^{-n} \otimes_B g^n. \tag{4.139}$$

By eq. (4.34), the translation map on P is given explictly as, for all $n \in \mathbb{Z}$,

$$\tau : H \to P \otimes_B P, \;\; u^n \mapsto \kappa^{-1}_{P\otimes_B P}(\tau_1(u^n), \tau_2(u^n)), \tag{4.140}$$

i.e., for all $n \in \mathbb{Z}$, the element $\tau(u^n) \in P \otimes_B P$ is uniquely determined by the property

$$(\chi_1 \otimes_B \chi_1)(\tau(u^n)) = h^{-n} \otimes_B h^n, \;\; (\chi_2 \otimes_B \chi_2)(\tau(u^n)) = g^{-n} \otimes_B g^n, \tag{4.141}$$

where maps χ_1, χ_2 were defined in (4.137). In order to find $\tau(u^n)$, first note that, for all $n \in \mathbb{Z}$,

$$(\chi_1 \otimes_B \chi_1)(b^n \otimes_B b^{-n}) = h^{-n} \otimes_B h^n, \;\; (\chi_2 \otimes_B \chi_2)(a^{-n} \otimes_B a^n) = g^{-n} \otimes_B g^n. \tag{4.142}$$

Then, for all $n \in \mathbb{Z}$,

$$\begin{aligned} b^n \otimes_B b^{-n} &= a^{-n}a^n b^n \otimes_B b^{-n} + (1 - a^{-n}a^n)b^n \otimes_B b^{-n} \\ &= a^{-n} \otimes_B a^n b^n b^{-n} + (1 - a^{-n}a^n)b^n \otimes_B b^{-n} \\ &= a^{-n} \otimes_B a^n + a^{-n} \otimes_B a^n(b^n b^{-n} - 1) + (1 - a^{-n}a^n)b^n \otimes_B b^{-n}. \end{aligned} \tag{4.143}$$

Observe that, for all $n \in \mathbb{Z}$, $1 - a^{-n}a^n \in \ker \chi_2$ and $b^n b^{-n} - 1 \in \ker \chi_1$, therefore the elements

$$\begin{aligned} \tau(u^n) &= b^n \otimes_B b^{-n} + a^{-n} \otimes_B a^n(1 - b^n b^{-n}) \\ &= a^{-n} \otimes_B a^n + (1 - a^{-n}a^n)b^n \otimes_B b^{-n}, \end{aligned} \tag{4.144}$$

$n \in \mathbb{Z}$, satisfy conditions (4.141), and hence define the translation map on P.

The above method of computation was inspired by the proof of Proposition 1 in [15].

4.7.4 Representations of $\vartheta(L_\beta^{p,q,\theta})$

To find representations of $\vartheta(L_\beta^{p,q,\theta})$ we use the same method as was used in [24]. Let $\varrho : \vartheta(L_\beta^{p,q,\theta}) \to \mathbf{B}(\mathcal{H})$ be any representation of $\vartheta(L_\beta^{p,q,\theta})$ as a subalgebra of the algebra of bounded operators $\mathbf{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$. Note that, by the relations (4.130), (4.126a) and (4.126c), the subspaces $\ker \varrho(\zeta)$ and $\ker \varrho(1 - zz^* - \zeta)$ are invariant. For any pair of closed subspaces $\mathcal{S} \subseteq \mathcal{S}' \subseteq \mathcal{H}$, let $\mathcal{S}^{\perp_{\mathcal{S}'}}$ denote the closure of the orthogonal complement of $\mathcal{S}$ in $\mathcal{S}'$. For brevity, we denote $\mathcal{S}^\perp = \mathcal{S}^{\perp_{\mathcal{H}}}$. In this section symbol '$\oplus$' denotes the orthogonal direct sum of Hilbert spaces. Hilbert space $\mathcal{H}$ can be decomposed into a direct sum

$$\begin{aligned} \mathcal{H} = \ker \varrho(\zeta) \oplus (\ker \varrho(\zeta))^\perp &= (\ker \varrho(\zeta) \cap \ker \varrho(1 - zz^* - \zeta)) \\ &\oplus (\ker \varrho(\zeta) \cap \ker \varrho(1 - zz^* - \zeta))^{\perp_{\ker \varrho(\zeta)}} \oplus (\ker \varrho(\zeta))^\perp. \end{aligned}$$

Suppose that $\Psi \in (\ker \varrho(\zeta))^\perp$ is such that $\varrho(1 - zz^* - \zeta)\Psi \neq 0$. Since $(\ker \varrho(\zeta))^\perp$ is an invariant subspace, we obtain, by the relation (4.126b),

$$0 = \varrho(\zeta(1 - zz^* - \zeta))\Psi = \varrho(\zeta)\varrho(1 - zz^* - \zeta)\Psi \neq 0,$$

which is a contradiction. It follows that $(\ker \varrho(\zeta))^{\perp} \subseteq \ker \varrho(1 - zz^* - \zeta)$. For brevity, let us denote $\mathcal{H}_0 = \ker \varrho(\zeta) \cap \ker \varrho(1 - zz^* - \zeta)$, $\mathcal{H}' = (\ker \varrho(\zeta) \cap \ker \varrho(1 - zz^* - \zeta))^{\perp_{\ker \varrho(\zeta)}}$, $\mathcal{H}'' = (\ker \varrho(\zeta))^{\perp}$. It follows that $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}' \oplus \mathcal{H}''$, and we have an orthogonal direct sum decomposition of the representation ϱ into subrepresentations

$$\varrho_0 : \vartheta(L_\beta^{p,q,\theta}) \to \mathbf{B}(\mathcal{H}_0), \ \ \varrho' : \vartheta(L_\beta^{p,q,\theta}) \to \mathbf{B}(\mathcal{H}'), \ \ \varrho'' : \vartheta(L_\beta^{p,q,\theta}) \to \mathbf{B}(\mathcal{H}'').$$

In the representation ϱ_0, the relations (4.126) are reduced to

$$\varrho_0(\zeta) = 0, \ \ \varrho_0(b) = \varrho_0(z)^\beta \varrho_0(a^*), \tag{4.145a}$$
$$\varrho_0(z^*)\varrho_0(z) = 1 = \varrho_0(z)\varrho_0(z^*), \ \ \varrho_0(a^*)\varrho_0(a) = 1 = \varrho_0(a)\varrho_0(a^*), \tag{4.145b}$$
$$\varrho_0(a)\varrho_0(z^{\pm 1}) = e^{\pm i\theta}\varrho_0(z^{\pm 1})\varrho_0(a). \tag{4.145c}$$

Similarly, in the representation ϱ', the relations (4.126) assume the form

$$\varrho'(\zeta) = 0, \ \ \varrho'(b) = \varrho'(z)^\beta \varrho'(a^*), \tag{4.146a}$$
$$\varrho'(z^*)\varrho'(z) - q\varrho'(z)\varrho'(z^*) = 1 - q, \ \ \varrho'(a^*)\varrho'(a) = 1 = \varrho'(a)\varrho'(a^*), \tag{4.146b}$$
$$\varrho'(a)\varrho'(z^{\pm 1}) = e^{\pm i\theta}\varrho'(z^{\pm 1})\varrho'(a). \tag{4.146c}$$

Finally, the representation ϱ'' reduces the relations (4.126) to the form

$$\varrho''(\zeta) = 1 - \varrho''(z)\varrho''(z^*), \ \ \varrho''(a) = \varrho''(b^*)\varrho''(z)^\beta, \tag{4.147a}$$
$$\varrho''(z^*)\varrho''(z) - p\varrho''(z)\varrho''(z^*) = 1 - p, \ \ \varrho''(b^*)\varrho''(b) = 1 = \varrho''(b)\varrho''(b^*), \tag{4.147b}$$
$$\varrho''(z)\varrho''(b^{\pm 1}) = e^{\pm i\theta}\varrho''(b^{\pm 1})\varrho''(z). \tag{4.147c}$$

From the representation theory of the quantum solid torus (Section 2.5.1), it follows that irreducible representations of $\vartheta(L_\beta^{p,q,\theta})$ include the ones unitarily equivalent to one of the following representations.

For all $0 \leq \mu < 2\pi$, there exists a representation $\varrho'_\mu : \vartheta(L_\beta^{p,q,\theta}) \to \mathbf{B}(\mathcal{H}'_\mu)$, where $\mathcal{H}'_\mu$ has an orthonormal Hilbert basis Ψ'_n, $n \in \mathbb{N}_0$, such that, for all $n \in \mathbb{N}_0$,

$$\begin{gathered}
\varrho'_\mu(z)\Psi'_n = \sqrt{1 - q^{n+1}}\Psi'_{n+1}, \ \ \varrho'_\mu(z^*)\Psi'_n = \sqrt{1 - q^n}\Psi'_{n-1} \text{ if } n > 0, \ \ \varrho'_\mu(z^*)\Psi'_0 = 0, \\
\varrho'_\mu(a^{\pm 1})\Psi'_n = e^{\pm i(\mu + n\theta)}\Psi'_n, \ \ \varrho'_\mu(\zeta)\Psi'_n = 0, \\
\varrho'_\mu(b)\Psi'_n = e^{-i(\mu + n\theta)}\sqrt{1 - q^{n+1}} \ldots \sqrt{1 - q^{n+\beta}}\Psi'_{n+\beta}, \\
\varrho'_\mu(b^*)\Psi'_n = \begin{cases} 0 & \text{if } n < \beta, \\ e^{i(\mu + (n-\beta)\theta)}\sqrt{1 - q^n} \ldots \sqrt{1 - q^{n-\beta+1}}\Psi'_{n-\beta} & \text{otherwise.} \end{cases}
\end{gathered} \tag{4.148}$$

Similarly, for all $0 \leq \mu < 2\pi$, there exists a representation $\varrho''_\mu : \vartheta(L_\beta^{p,q,\theta}) \to \mathbf{B}(\mathcal{H}''_\mu)$, where $\mathcal{H}''_\mu$ has an orthonormal Hilbert basis Ψ'_n, $n \in \mathbb{N}_0$, such that, for all

$n \in \mathbb{N}_0$,

$$\varrho''_\mu(z)\Psi''_n = \sqrt{1-p^{n+1}}\Psi''_{n+1},\ \varrho''_\mu(z^*)\Psi''_n = \sqrt{1-p^n}\Psi''_{n-1} \text{ if } n>0,\ \varrho''_\mu(z^*)\Psi''_0 = 0,$$
$$\varrho''_\mu(b^{\pm 1})\Psi''_n = e^{\pm i(\mu-n\theta)}\Psi''_n,\ \varrho''_\mu(\zeta)\Psi''_n = p^n\Psi''_n,$$
$$\varrho''_\mu(a)\Psi''_n = e^{-i(\mu-(n+\beta)\theta)}\sqrt{1-p^{n+1}}\dots\sqrt{1-p^{n+\beta}}\Psi''_{n+\beta},$$
$$\varrho''_\mu(a^*)\Psi''_n = \begin{cases} 0 & \text{if } n<\beta, \\ e^{i(\mu-n\theta)}\sqrt{1-p^n}\dots\sqrt{1-p^{n-\beta+1}}\Psi''_{n-\beta} & \text{otherwise.} \end{cases} \tag{4.149}$$

Finally, depending on whether deformation parameter θ is a rational or irrational multiple of 2π, we have one of the following families of irreducible representations.

Suppose that $\theta = 2\pi\frac{M}{N}$, where $M, N \in \mathbb{Z}$, $N > 0$, and M and N are relatively prime. Then, for all $0 \leq \mu, \nu < 2\pi$, there exists a representation $\varrho_0^{\mu\nu} : \vartheta(L_\beta^{p,q,\theta}) \to \mathbf{B}(\mathcal{H}_0^{\mu\nu})$, where $\mathcal{H}_0^{\mu\nu}$ has an orthonormal Hilbert basis Ψ_n, $n \in \mathbb{Z}_N$, and, for all $n \in \mathbb{Z}_N$,

$$\varrho_0^{\mu\nu}(a^{\pm 1})\Psi_n = e^{\pm i\frac{\mu}{N}\pm in\theta}\Psi_n,\ \varrho_0^{\mu\nu}(z^{\pm 1})\Psi_n = e^{\pm i\frac{\nu}{N}}\Psi_{n\pm 1},\ \varrho_0^{\mu\nu}(\zeta)\Psi_n = 0,$$
$$\varrho_0^{\mu\nu}(b)\Psi_n = e^{i\frac{\nu\beta-\mu}{N}-in\theta}\Psi_{n+\beta},\ \varrho_0^{\mu\nu}(b^*)\Psi_n = e^{i\frac{\mu-\nu\beta}{N}+i(n-\beta)\theta}\Psi_{n-\beta}. \tag{4.150}$$

If θ is an irrational multiple of 2π, then, for any $0 \leq \mu < 2\pi$, we have a representation $\varrho_0^\mu : \vartheta(L_\beta^{p,q,\theta}) \to \mathbf{B}(\mathcal{H}_0^\mu)$. The Hilbert space $\mathcal{H}_0^\mu$ has an orthonormal basis Ψ_n, $n \in \mathbb{Z}$. For all $n \in \mathbb{Z}$,

$$\varrho_0^\mu(a^{\pm 1})\Psi_n = e^{\pm i\mu\pm in\theta}\Psi_n,\ \varrho_0^\mu(z^{\pm 1})\Psi_n = \Psi_{n\pm 1},\ \varrho_0^\mu(\zeta)\Psi_n = 0,$$
$$\varrho_0^\mu(b)\Psi_n = e^{-i\mu-in\theta}\Psi_{n+\beta},\ \varrho_0^\mu(b^*)\Psi_n = e^{i\mu+i(n-\beta)\theta}\Psi_{n-\beta}. \tag{4.151}$$

The coordinate algebra $\vartheta(L_\beta^{p,q,\theta})$ can be completed to the enveloping C*-algebra $C(L_\beta^{p,q,\theta})$ with the norm

$$\|w\| = \sup_\varrho \|w\|_\varrho,\ w \in \vartheta(L_\beta^{p,q,\theta}), \tag{4.152}$$

where supremum is taken over all bounded representations $\varrho : \vartheta(L_\beta^{p,q,\theta}) \to \mathbf{B}(\mathcal{H})$, and $\|\cdot\|_\varrho$ denotes the operator norm in representation ϱ.

4.7.5 Final remarks

We conclude this section with a number of remarks about the structure of quantum lens spaces. We also comment on the K-theory of quantum lens spaces. The detailed developments of the topics discussed here, can be considered as directions for future work.

The relations (4.126) defining $\vartheta(L_\beta^{p,q,\theta})$ assume a particularly simple form in the case $\beta = 1$. Namely, if $\beta = 1$, then $z = ba$, $\xi = 1 - aa^*$, and the relations (4.126) are reduced to

$$a^*a - paa^* = 1 - p, \quad b^*b - qbb^* = 1 - q, \quad ab = e^{i\theta}ba, \quad ab^* = e^{-i\theta}b^*a,$$
$$(1 - aa^*)(1 - bb^*) = 0. \tag{4.153}$$

It follows that the quantum lens space of charge 1, $L_1^{p,q,\theta}$, can be identified with the Heegaard quantum sphere $S^3_{p,q,\theta}$ considered in [2].

For brevity, let us denote $A = \vartheta(S^3_{p,q,\theta})$. Let us define a $\mathbb{Z}$-grading on A with

$$\deg(a) = 1, \quad \deg(a^*) = -1, \quad \deg(b) = -1, \quad \deg(a^*) = 1, \tag{4.154}$$

and, for all $n \in \mathbb{Z}$, let $A_n = \{w \in A \mid \deg(w) = n\}$. For any $\beta \in \mathbb{N}$, define the subalgebra $A(\beta) = \bigoplus_{n\in\mathbb{Z}} A_{\beta n}$. It can be shown that the $*$-algebra map $f_\beta : \vartheta(L_\beta^{p,q,\beta\theta}) \to A(\beta)$, given on generators by

$$f_\beta(\xi) = 1 - aa^*, \quad f_\beta(z) = ba, \quad f_\beta(a) = a^\beta, \quad f_\beta(b) = e^{i\theta\frac{\beta(\beta-1)}{2}}b^\beta, \tag{4.155}$$

is a well defined $*$-algebra isomorphism. It can be demonstrated that the subalgebra $A(\beta)$ corresponds classically to the quotient of the three sphere by the $\mathbb{Z}_\beta$-action (2.36).

In [2], it was demonstrated that $C(S^3_{p,q,\theta})$ is isomorphic as a C^*-algebra to a fibre product of two C^*-algebras isomorphic to quantum solid tori, and then the Mayer-Vietoris sequence was used to compute the K-theory of $C(S^3_{p,q,\theta})$ using the K-theory of quantum solid tori. We expect that this method can be adapted to compute the K-theory of $C(L_\beta^{p,q,\theta})$. This is a direction for future work.

Bibliography

[1] W. Arveson. An invitation to C*-algebras. Graduate Texts in Mathematics, No. 39. *Springer-Verlag, New York-Heidelberg*, 1976. x+106 pp.

[2] P. Baum, P. M. Hajac, R. Matthes, and W. Szymański. The K-theory of Heegaard-type quantum 3-spheres. To appear in *K-theory*, arXiv:math.KT/0409573.

[3] T. Brzeziński. On modules associated to coalgebra Galois extensions. *J. Algebra*, 215(1):290–317, 1999.

[4] T. Brzeziński. Galois comodules. *J. Algebra*, 290:503–537, 2005.

[5] T. Brzeziński, L. Dąbrowski, and B. Zieliński. Hopf fibration and monopole connection over the contact quantum spheres. *J. Geom. Phys.*, 50(1-4):345–359, 2004.

[6] T. Brzeziński and P. M. Hajac. Coalgebra extensions and algebra coextensions of Galois type. *Comm. Algebra*, 27(3):1347–1367, 1999.

[7] T. Brzeziński and P. M. Hajac. The Chern-Galois character. *C. R. Math. Acad. Sci. Paris*, 338(2):113–116, 2004.

[8] T. Brzeziński, P. M. Hajac, R. Matthes, and W. Szymański. The Chern character for principal extensions of noncommutative algebras. http://www.fuw.edu.pl/ pmh/artpdf/bhms.pdf.

[9] T. Brzeziński and S. Majid. Quantum group gauge theory on quantum spaces. *Comm. Math. Phys.*, 157(3):591–638, 1993. Erratum — 167, 235, 1995.

[10] T. Brzeziński and S. Majid. Coalgebra bundles. *Comm. Math. Phys.*, 191(2):467–492, 1998.

[11] T. Brzeziński and S. Majid. Quantum geometry of algebra factorisations and coalgebra bundles. *Comm. Math. Phys.*, 213(3):491–521, 2000.

[12] T. Brzeziński and A. Sitarz. Dirac monopoles from the Matsumoto noncommutative spheres. In *Group24: Physical and Mathematical Aspects of Symmetries*, volume 173 of *Conference Series*, pages 791–794. J-P Gazean et al. (eds) IOP, Bristol, 2003.

[13] T. Brzeziński and R. Wisbauer. *Corings and Comodules.* Cambridge University Press, 2003.

[14] R. J. Budzyński and W. Kondracki. Quantum principal fibre bundles: topological aspects. *Rep. Math. Phys.*, 37(3):365–385, 1996.

[15] D. Calow and R. Matthes. Covering and gluing of algebras and differential algebras. *J. Geom. Phys.*, 32(4):364–396, 2000.

[16] D. Calow and R. Matthes. Connections on locally trivial quantum principal fibre bundles. *J. Geom. Phys.*, 41(1-2):114–165, 2002.

[17] D. Calow and R. Matthes. Locally trivial quantum vector bundles and associated vector bundles. arXiv:math.QA/0002229.

[18] S. U. Chase and M. E. Sweedler. *Hopf algebras and Galois theory.* Lecture Notes in Mathematics, Vol. 97. Springer-Verlag, Berlin, 1969.

[19] A. Connes. *Noncommutative geometry.* Academic Press Inc., San Diego, CA, 1994.

[20] L. Dąbrowski. The garden of quantum spheres. In *Noncommutative geometry and quantum groups (Warsaw, 2001)*, volume 61 of *Banach Center Publ.*, pages 37–48. Polish Acad. Sci., Warsaw, 2003.

[21] J. Dieudonné. *Éléments d'analyse. Tome I.* Cahiers Scientifiques [Scientific Reports], XXVIII. Gauthier-Villars, Paris, 1981.

[22] L. El Kaoutit and J. Gómez-Torrecillas. Comatrix corings: Galois corings, descent theory, and a structure theorem for cosemisimple corings. *Math. Z.*, 244(4):887–906, 2003.

[23] P. M. Hajac. Strong connections on quantum principal bundles. *Comm. Math. Phys.*, 182(3):579–617, 1996.

[24] P. M. Hajac, R. Matthes, and W. Szymański. A locally trivial quantum Hopf fibration. To appear in *Algebr. Represent. Theory*, arXiv:math.QA/0112317.

[25] P. M. Hajac, R. Matthes, and W. Szymański. Quantum real projective space, disc and spheres. *Algebr. Represent. Theory*, 6(2):169–192, 2003.

[26] C. Kassel and H.-J. Schneider. Homotopy theory of Hopf Galois extensions. Preprint arXiv:math.QA/0402034, 2004.

[27] S. Klimek and A. Lesniewski. Quantum Riemann surfaces. I. The unit disc. *Comm. Math. Phys.*, 146(1):103–122, 1992.

[28] S. Klimek and A. Lesniewski. A two-parameter quantum deformation of the unit disc. *J. Funct. Anal.*, 115(1):1–23, 1993.

[29] H. F. Kreimer and M. Takeuchi. Hopf algebras and Galois extensions of an algebra. *Indiana Univ. Math. J.*, 30(5):675–692, 1981.

[30] K. Matsumoto. Noncommutative three-dimensional spheres. *Japan. J. Math. (N.S.)*, 17(2):333–356, 1991.

[31] K. Matsumoto and J. Tomiyama. Noncommutative lens spaces. *J. Math. Soc. Japan*, 44(1):13–41, 1992.

[32] G. J. Murphy. *C^*-algebras and operator theory*. Academic Press Inc., Boston, MA, 1990.

[33] H. Omori, Y. Maeda, N. Miyazaki, and A. Yoshioka. Noncommutative 3-sphere: a model of noncommutative contact algebras. *J. Math. Soc. Japan*, 50(4):915–943, 1998.

[34] M. J. Pflaum. Quantum groups on fibre bundles. *Comm. Math. Phys.* 166 (1994), no. 2, 279–315.

[35] P. Podleś. Quantum spheres. *Lett. Math. Phys.*, 14(3):193–202, 1987.

[36] M. A. Rieffel. C^*-algebras associated with irrational rotations. *Pacific J. Math.* 93 (1981), no. 2, 415–429.

[37] P. Schauenburg. Bi-Galois objects over the Taft algebras. *Israel J. Math.*, 115:101–123, 2000.

[38] H. J. Schneider. Principal homogenous spaces for arbitrary Hopf algebras. *Israel J. Math.*, 72(1-2), 1990.

[39] H. J. Schneider. Representation theory of Hopf Galois extensions. *Israel J. Math.*, 72(1-2):196–231, 1990.

[40] A. J. Sheu. Quantization of the Poisson SU(2) and its Poisson homogeneous space – the 2-sphere. *Comm. Math. Phys.* 135 (1991), no. 2, 217–232.

[41] M. Takeuchi. A note on geometrically reductive groups. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 20:387–396, 1973.

[42] M. Takeuchi. On extensions of formal groups by μ^A. *Comm. Algebra*, 5(13):1439–1481, 1977.

[43] T. T. Wu and C. M. Yang. Concept of nonintegrable phase factors and global formulation of gauge fields. *Phys. Rev*, D12:3845–3857, 1975.